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VARIATIONS OF THE TELESCOPE CONJECTURE AND BOUSFIELD LATTICES FOR LOCALIZED CATEGORIES OF SPECTRA

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Abstract. We investigate several versions of the telescope conjecture on localized categories of spectra, and implications between them. Generalizing the "finite localization" construction, we show that on such categories, localizing away from a set of strongly dualizable objects is smashing. We classify all smashing localizations on the harmonic category, $H\mathbb{F}_p$-local category and I-local category, where I is the Brown-Comenetz dual of the sphere spectrum; all are localizations away from strongly dualizable objects, although these categories have no nonzero compact objects. The Bousfield lattices of the harmonic, $E(n)$-local, $K(n)$-local, $H\mathbb{F}_p$-local and I-local categories are described, along with some lattice maps between them. One consequence is that in none of these categories is there a nonzero object that squares to zero. Another is that the $H\mathbb{F}_p$-local category has localizing subcategories that are not Bousfield classes.

1. Introduction

The telescope conjecture, first stated by Ravenel [Rav84, Conj. 10.5], is a claim about two classes of localization functors in the p-local stable homotopy category of spectra. First, one can localize away from a finite type $n+1$ spectrum $F(n+1)$; the acyclics are the smallest localizing subcategory containing $F(n+1)$, and we denote this functor L_n^f. Second, one can localize at the wedge of the first $n+1$ Morava K-theories $K(0) \vee \cdots \vee K(n)$; the acyclics are all spectra that smash with $K(0) \vee \cdots \vee K(n)$ to zero and this is denoted L_n. Both these localizations are smashing, i.e. they commute with coproducts. The telescope conjecture (TC_n), basically, claims that L_n^f and L_n are isomorphic. In fact, here we consider three slightly different versions, $\mathsf{TC1_n}$, $\mathsf{TC2_n}$, and $\mathsf{TC3_n}$, of the telescope conjecture. In Section 3 we articulate them carefully and show implications between them.

The conjecture is known to hold for $n = 0$ [Rav92, p. 79], and for $n = 1$ when $p = 2$ [Mah82] and $p > 2$ [Mil81]. A valiant but unsuccessful effort at a counterexample, for $n \geq 2$, was undertaken by Mahowald, Ravenel, and Shick, as outlined in [MRS01]. Since then little progress has been made, and the original conjecture remains open.

A generalization of the telescope conjecture can be stated for spectra, as well as other triangulated categories. Localization away from a finite spectrum, i.e. a

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compact object of the category, always yields a smashing localization functor (see e.g. [Bou79, Prop. 2.9] or [Mil92] or [HPS97, Thm. 3.3.3]). The Generalized Smashing Conjecture (GSC) is that every smashing localization arises in this way. If true, then every smashing localization is determined by its compact acyclics; if the GSC holds in spectra, then so must the TC_n for all n.

The GSC, essentially stated for spectra decades ago by Bousfield [Bou79b, Conj. 3.4], has been formulated in many other triangulated categories, in many cases labeled as the telescope conjecture, and in many cases proven to hold. Neeman [Nee92] made the conjecture for the derived category $D(R)$ of a commutative ring R, and showed it holds when the ring is Noetherian. See also [HPS97, Thm. 6.3.7] or [KS10] for a generalization. On the other hand, Keller [Kel94] gave an example of a non-Noetherian ring for which the GSC fails. Benson, Iyengar, and Krause have shown that the GSC holds in a stratified category [BIK11], such as the stable module category of a finite group [BIK11b]. Balmer and Favi [BF11] show that in a tensor triangulated category with a good notion of support, the GSC is a "local" question.

It is worth noting that there are further variations of the GSC, which we won't consider here. Krause [Kra00] formulated a variation of the GSC, in terms of subcategories generated by sets of maps, that makes sense (and holds) for any compactly generated triangulated category. Krause and Solberg give a variation for stable module categories, stated in terms of cotorsion pairs [KS03]. See also [Kra05, AHST08, Brü07, Sto10].

To date, Keller's ring yields the only category where the GSC is known to fail. In this paper we give several more examples. Incidentally, each is a well generated triangulated category that is not compactly generated.

One of our main results is the following. We weaken the assumptions for "finite localization", and show that in many categories, localization away from any set of strongly dualizable objects yields a smashing localization. (Recall that an object X is strongly dualizable if $F(X, Y) \cong F(X, \mathbb{1}) \wedge Y$ for all Y, where $\mathbb{1}$ is the tensor unit and $F(-,-)$ the function object bifunctor.) Let $\mathsf{loc}(X)$ denote the smallest localizing subcategory containing X. We prove the following as Theorem 3.5.

Theorem A: Let T be a well generated tensor triangulated category such that $\mathsf{loc}(\mathbb{1}) = \mathsf{T}$. Let $A = \{B_\alpha\}$ be a (possibly infinite) set of strongly dualizable objects. Then there exists a smashing localization functor $L : \mathsf{T} \to \mathsf{T}$ with Ker $L = \mathsf{loc}(A)$.

Thus we are led to conjecture the following.

Strongly Dualizable Generalized Smashing Conjecture (SDGSC)**:** Every smashing localization is localization away from a set of strongly dualizable objects.

We give several examples of categories where the GSC fails, but the SDGSC holds. In fact, we consider a topological setting, where one can also formulate a version (or versions, rather) of the original telescope conjecture.

Specifically, we consider localized categories of spectra. Let $\mathcal{S}$ be the p-local stable homotopy category, and let $\wedge$ denote the smash, i.e. tensor, product. Take Z an object of $\mathcal{S}$, and let $L = L_Z : \mathcal{S} \to \mathcal{S}$ be the localization functor that annihilates Z_*-acyclic objects. The full subcategory of L-local objects, that is, objects X for which $X \to LX$ is an equivalence, has a tensor triangulated structure induced by that of $\mathcal{S}$. Let $\mathcal{L}$ denote this category; the triangles are the same as in $\mathcal{S}$, the coproduct is $X \coprod_{\mathcal{L}} Y = L(X \coprod Y)$ and the tensor is $X \wedge_{\mathcal{L}} Y = L(X \wedge Y)$.

In Definition 3.6, we define localization functors l_n^f and l_n on $\mathcal{L}$ that are localized versions of L_n^f and L_n. The localized telescope conjecture (LTC), basically, is that l_n^f and l_n are isomorphic. In fact, we give three versions of the LTC, and in Theorems 3.12 and 3.13 establish implications between them. Then, examining specific examples of localized categories of spectra, we conclude the following in Theorems 4.3, 5.11, 6.1, 6.5 and 6.9 and Corollary 5.6.

Theorem B: All versions of the localized telescope conjecture, $\mathsf{LTC1_i}$, $\mathsf{LTC2_i}$, and $\mathsf{LTC3_i}$ hold for all $i \geq 0$, in the $\bigvee_{n\geq 0} K(n)$-local (i.e. harmonic), $K(n)$-local, $H\mathbb{F}_p$-local, BP-local, and I-local categories, where I is the Brown-Comenetz dual of the sphere spectrum.

In order to consider the GSC and SDGSC in $\mathcal{L}$, we must classify the smashing localizations on $\mathcal{L}$. We are able to do this in several examples.

Theorem C: In the harmonic category, the GSC fails but the SDGSC holds. Likewise in the $H\mathbb{F}_p$-local and I-local categories. In the BP-local category the GSC fails but the SDGSC is open. In the $E(n)$-local and $K(n)$-local categories the GSC and SDGSC both hold.

Proof. This is Theorems 4.4 and 5.11, Propositions 6.3 and 6.10, and Corollaries 5.6 and 6.7. □

One novelty in our approach is our use of Bousfield lattice arguments. Given an object X in a tensor triangulated category T, the Bousfield class of X is $\langle X \rangle = \{W \mid W \wedge X = 0\}$. It is now known [IK13] that every well generated tensor triangulated category has a set of Bousfield classes. This set has the structure of a lattice, and is called the Bousfield lattice of T. One can now attempt to calculate the Bousfield lattices of categories of localized spectra. Furthermore, every smashing localization yields a pair of so-called complemented Bousfield classes. Information about the Bousfield lattice of a category gives information about its complemented classes, which gives information about the smashing localization functors on the category.

Moreover, the first version of the telescope conjecture $\mathsf{TC1_n}$ is that two spectra $T(n)$ and $K(n)$ have the same Bousfield class. In the localized version this becomes ($\mathsf{LTC1_n}$) the claim that $\langle LT(n)\rangle = \langle LK(n)\rangle$ in the Bousfield lattice of $\mathcal{L}$. One is thus led to investigating Bousfield lattices of localized spectra.

Corollary 2.7 gives an upper bound, $2^{2^{\aleph_0}}$, on the cardinality of such lattices. Jon Beardsley has calculated the Bousfield lattice of the harmonic category to be isomorphic to the power set of $\mathbb{N}$; we give this calculation in Proposition 4.2. In Corollary 5.4 and Proposition 5.7 we show that one can realize this lattice as an inverse limit of the Bousfield lattices of $E(n)$-local categories, as n ranges over $\mathbb{N}$. Then in Corollary 5.10 and Propositions 6.2 and 6.6, we show that the $K(n)$-local, $H\mathbb{F}_p$-local, and I-local categories all have two-element Bousfield lattices. In Proposition 6.11 we give a lower bound, $2^{\aleph_0}$, on the cardinality of the Bousfield lattice of the BP-local category.

One immediate object-level application of these Bousfield lattice calculations is the following. Call an object X square-zero if it is nonzero, but $X \wedge X = 0$. Then Proposition 2.9 shows that there are no square-zero objects in the harmonic, $E(n)$-, $K(n)$-, $H\mathbb{F}_p$-, or I-local categories.

We are also able to answer the analogue of a conjecture by Hovey and Palmieri, originally stated for the stable homotopy category. Conjecture 9.1 in [HP99] is that every localizing subcategory is a Bousfield lattice. Proposition 6.4 demonstrates that this fails in the $H\mathbb{F}_p$-local category, by giving two localizing subcategories that are not Bousfield classes.

Section 2 establishes the categorical setting, and provides background on localization, Bousfield lattices, and stable homotopy theory. Section 3 defines the various versions of the telescope conjecture, for spectra and for localized spectra, and establishes implications among them. The remainder of the paper is devoted to examining specific examples: the harmonic category (Section 4), the $E(n)$-local and $K(n)$-local categories (Section 5), and the $H\mathbb{F}_p$-local, I-local, BP-local, and $F(n)$-local categories (Section 6). All results are new unless cited. Most of the results on the $E(n)$-local and $K(n)$-local categories in Section 5 follow in a straightforward way from Hovey and Strickland's work in [HS99], and are included for completeness.

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2. Preliminaries

2.1. **Categorical setting.** We start with the notion of a tensor triangulated category C; i.e. a triangulated category with set-indexed coproducts and a closed symmetric monoidal structure compatible with the triangulation [HPS97, App.A]. Let $\Sigma : \mathsf{C} \to \mathsf{C}$ denote the shift, $[X, Y]$ the morphisms from X to Y, and $[X, Y]_n = [\Sigma^n X, Y]$ for any $n \in \mathbb{Z}$.

Let $- \wedge -$ denote the smash (tensor) product, $\mathbb{1}$ denote the unit, and $F(-, -)$ denote the function object bifunctor; $F(X, -)$ is the right adjoint to $X \wedge -$. Recall that an object X in C is said to be *strongly dualizable* if the natural map $DX \wedge Y \to F(X, Y)$ is an isomorphism for all Y, where $DX = F(X, \mathbb{1})$ is the Spanier-Whitehead dual. Since $F(\mathbb{1}, X) \cong X$ for all X, the map $F(\mathbb{1}, \mathbb{1}) \wedge Y \to F(\mathbb{1}, Y)$ is an equivalence and $\mathbb{1}$ is always strongly dualizable.

For a regular cardinal α, we say an object X is *α-small* if every morphism $X \to \coprod_{i \in I} Y_i$ factors through $\coprod_{i \in J} Y_i$ for some $J \subseteq I$ with $|J| < \alpha$. If X is $\aleph_0$-small we say X is *compact* ([HPS97] calls this *small*); this is equivalent to the condition that the natural map $\oplus_{i \in K}[X, Z_i] \to [X, \coprod_{i \in K} Z_i]$ is an isomorphism for any set-indexed coproduct $\coprod_{i \in K} Z_i$. We say C is *α-well generated* if it has a set of perfect generators [Kra10, Sect. 5.1] which are α-small, and C is *well generated* if it is α-well generated for some α. See [Kra10] for more details.

A *localizing* subcategory is a triangulated subcategory of C that is closed under retracts and coproducts; a *thick* subcategory is a triangulated subcategory that is closed under retracts. Given an object or set of objects X, let $\mathsf{loc}(X)$ (resp. $\mathsf{th}(X)$)

denote the smallest localizing (resp. thick) subcategory containing X. We say that $\mathsf{loc}(X)$ is *generated by* X.

Definition/Notation 2.1. Throughout this paper let T be a well generated tensor triangulated category such that $\mathsf{loc}(\mathbb{1}) = \mathsf{T}$.

In the language of [HPS97], such a T is almost a "monogenic stable homotopy category", except that we do not insist that the unit $\mathbb{1}$ is compact.

In practice, in this paper T will always be either the p-local stable homotopy category of spectra $\mathcal{S}$ or the category $\mathcal{L}_Z$ of L_Z-local objects derived from a localization functor $L_Z : \mathcal{S} \to \mathcal{S}$. The former satisfies Definition 2.1 by [HPS97, Ex. 1.2.3(a)], and the latter by Theorem 2.3 and Lemma 2.4 below.

2.2. **Background on localization.** Recall that a *localization functor* (or simply *localization*) on a tensor triangulated category C is an exact functor $L : \mathsf{C} \to \mathsf{C}$, along with a natural transformation $\eta : 1 \to L$ such that $L\eta$ is an equivalence and $L\eta = \eta L$. We call Ker L the *L-acyclics*. It follows that there is an exact functor $C : \mathsf{C} \to \mathsf{C}$, called *colocalization*, such that every X in C fits into an exact triangle $CX \to X \to LX$, with CX L-acyclic. An object Y is *L-local* if it is in the essential image of L, and this is equivalent to satisfying $[Z, Y] = 0$ for all L-acyclic Z. See [HPS97, Ch. 3] or [Kra10] for further background.

We also recall two special types of localizations. A localization $L : \mathsf{C} \to \mathsf{C}$ is said to be *smashing* if L preserves coproducts, equivalently if $LX \cong L\mathbb{1} \wedge X$ for all X.

Given a set A of objects of C, we say that a localization functor $L : \mathsf{C} \to \mathsf{C}$ is *localization away from A* if the L-acyclics are precisely $\mathsf{loc}(A)$. If such a localization exists, we also say it is *generated by A*. When $\mathsf{C} = \mathcal{S}$, it is well known (e.g. [Mil92, MS95]) that localization away from a set of compact objects exists, and yields a smashing localization functor. As mentioned in the introduction, this result has been generalized to other categories as well (e.g. [HPS97, Thm. 3.3.3], [BF11, Thm. 4.1]). We present a further generalization in Theorem 3.5.

In this paper we will restrict our attention to homological localizations, which we now describe. Given an object Z in a tensor triangulated category C, the *Bousfield class* of Z is defined to be

$$\langle Z \rangle = \{W \in \mathsf{C} \mid W \wedge Z = 0\}.$$

Extending a classical result of Bousfield's for $\mathcal{S}$, Iyengar and Krause recently showed [IK13, Prop. 2.1] that for every object Z in a well generated tensor triangulated category C, there is a localization functor $L_Z : \mathsf{C} \to \mathsf{C}$ with L_Z-acyclics precisely $\langle Z \rangle$. We call such an L_Z *homological localization at Z*.

Notation 2.2. Let T be as in Definition 2.1, with tensor unit $\mathbb{1}$. For an object Z in T, let $L_Z : \mathsf{T} \to \mathsf{T}$ be homological localization at Z. Let $\mathcal{L}_Z$ denote the category of L_Z-local objects, the essential image of L_Z.

Theorem 2.3. [HPS97, 3.5.1,3.5.2] Let $L = L_Z : \mathsf{T} \to \mathsf{T}$ be a localization, and $\mathcal{L}_Z$ the category of L_Z-local objects. Then $\mathcal{L}_Z$ has a natural structure as a tensor triangulated category, generated by $L_Z\mathbb{1}$, which is the unit. Considered as a functor from T to $\mathcal{L}_Z$, L preserves triangles, the tensor product and its unit, coproducts, and strong dualizability. Furthermore, L preserves compactness if and only if L is a smashing localization.

Explicitly, for L-local objects X, X_i and Y, in $\mathcal{L}$ we have $\coprod_{\mathcal{L}} X_i = L(\coprod_{\mathsf{T}} X_i)$ and $X \wedge_{\mathcal{L}} Y = L(X \wedge_{\mathsf{T}} Y)$ and $F_{\mathcal{L}}(X,Y) = F(X,Y)$. Note that $L_Z\mathbb{1}$ is strongly dualizable but may not be compact in $\mathcal{L}_Z$.

Lemma 2.4. *The category $\mathcal{L}_Z$ is well generated.*

Proof. By Proposition 2.1 of [IK13], the L_Z-acyclics $\langle Z\rangle$ form a well generated localizing subcategory of T. Then by [Kra10, Thm. 7.2.1], the Verdier quotient $\mathsf{T}/\langle Z\rangle$, which is equivalent to the local category $\mathcal{L}_Z$, is well generated. □

We conclude this subsection with a lemma with four useful well-known facts. Recall that a *ring object* in a tensor triangulated category is an object R with an associative multiplication map $\mu : R \wedge R \to R$ and a unit $\iota : \mathbb{1} \to R$, making the evident diagrams commute. If R is a ring object, then an *R-module object* is an object M with a map $m : R \wedge M \to M$ along with evident commutative diagrams. Note that $R \wedge X$ is an R-module object for every X. A *skew field object* is a ring object such that every R-module object is *free*, i.e. isomorphic to a coproduct of suspensions of R [HPS97, Def. 3.7.1].

Lemma 2.5. *Let C be a tensor triangulated category with $\mathsf{loc}(\mathbb{1}) = \mathsf{C}$, and $L : \mathsf{C} \to \mathsf{C}$ a localization.*

(1) *Every localizing subcategory S of C is tensor-closed; that is, given $X \in \mathsf{S}$ and $Y \in \mathsf{C}$, $X \wedge Y \in \mathsf{S}$.*
(2) *For all X and Y in C, $L(X \wedge Y) = L(LX \wedge LY)$.*
(3) *Considered as a functor from C to $\mathcal{L}$, L also preserves ring objects and module objects.*
(4) *If R is a ring object and M is an R-module object (in particular, if $M = R$), then M is R-local.*

Proof. For (1), note that $Y \in \mathsf{loc}(\mathbb{1}) = \mathsf{C}$, so $X \wedge Y \in \mathsf{loc}(X \wedge \mathbb{1}) = \mathsf{loc}(X) \subseteq \mathsf{S}$.

For (2), consider the exact triangle $X \wedge CY \to X \wedge Y \to X \wedge LY$. Since CY is L-acyclic and these form a localizing subcategory, $L(X \wedge CY) = 0$, so $L(X \wedge Y) = L(X \wedge LY)$. Using the same reasoning with the triangle $CX \wedge LY \to X \wedge LY \to LX \wedge LY$, the result follows.

If $R \in \mathsf{C}$, is a ring object, then $L(\mu) : L(R \wedge R) = L(LR \wedge LR) = LR \wedge_{\mathcal{L}} LR \to LR$, and all the localized diagrams commute. Similarly for module objects.

Part (4) is [Rav84, Prop. 1.17(a)]. □

2.3. **Background on Bousfield lattices.** Every well generated tensor triangulated category, and hence every localized category of spectra, has a set (rather than a proper class) of Bousfield classes [IK13, Thm. 3.1]. This was also recently shown for every tensor triangulated category with a combinatorial model [CGR14]. This set is called the *Bousfield lattice* $\mathsf{BL}(\mathsf{T})$ and has a lattice structure which we now recall. Refer to [HP99, Wol13] for more details.

The partial ordering is given by reverse inclusion: we say $\langle X\rangle \leq \langle Y\rangle$ when $W \wedge Y = 0 \implies W \wedge X = 0$. It is also helpful to remember that, unwinding definitions, $\langle X\rangle \leq \langle Y\rangle$ precisely when every L_X-local object is also L_Y-local. Clearly $\langle 0\rangle$ is the minimum and $\langle \mathbb{1}\rangle$ is the maximum class. The join of any set of classes is $\bigvee_{i\in I}\langle X_i\rangle = \langle \coprod_{i\in I} X_i\rangle$, and the meet is defined to be the join of all lower bounds.

The smash product induces an operation on Bousfield classes, where $\langle X\rangle \wedge \langle Y\rangle = \langle X \wedge Y\rangle$. This is a lower bound, but in general not the meet. However, if we restrict to the subposet $\mathsf{DL} = \{\langle W\rangle \mid \langle W \wedge W\rangle = \langle W\rangle\}$, then the meet and smash agree. Since coproducts distribute across the smash product, DL is a distributive lattice.

We say a class $\langle X\rangle$ is *complemented* if there exists a class $\langle X^c\rangle$ such that $\langle X\rangle \wedge \langle X^c\rangle = \langle 0\rangle$ and $\langle X\rangle \vee \langle X^c\rangle = \langle \mathbb{1}\rangle$. The collection of complemented classes is denoted BA. For example, every smashing localization $L : \mathsf{T} \to \mathsf{T}$ gives a pair of complemented classes, namely $\langle C\mathbb{1}\rangle$ and $\langle L\mathbb{1}\rangle$. Because every complemented class is also in DL, BA is a Boolean algebra.

Proposition 2.6. Let T be as in Definition 2.1, and $L_Z : \mathsf{T} \to \mathsf{T}$ a localization functor as in Notation 2.2. Then L_Z induces a well-defined order-preserving map of lattices $\mathsf{BL}(\mathsf{T}) \to \mathsf{BL}(\mathcal{L}_Z)$, where $\langle X\rangle \mapsto \langle L_Z X\rangle$. This map is surjective, and sends $\mathsf{DL}(\mathsf{T})$ onto $\mathsf{DL}(\mathcal{L}_Z)$ and $\mathsf{BA}(\mathsf{T})$ onto $\mathsf{BA}(\mathcal{L}_Z)$.

Proof. Most of this was proved in Lemma 3.1 of [Wol13]. For $\langle X\rangle \in \mathsf{DL}(\mathsf{T})$, using Lemma 3.10 we get

$$\langle LX\rangle = \langle L(X \wedge X)\rangle = \langle L(LX \wedge LX)\rangle = \langle LX \wedge_{\mathcal{L}} LX\rangle.$$

Likewise, one can check that for $\langle X\rangle \in \mathsf{BA}(\mathsf{T})$, the class $\langle LX\rangle \in \mathsf{BL}(\mathcal{L})$ is complemented by $\langle L(X^c)\rangle$, keeping in mind that $\langle L\mathbb{1}\rangle$ is the top class in $\mathsf{BL}(\mathcal{L})$. □

Corollary 2.7. For any $Z \in \mathcal{S}$, we have $|\mathsf{BL}(\mathcal{L}_Z)| \leq 2^{2^{\aleph_0}}$.

Proof. $|\mathsf{BL}(\mathcal{L}_Z)| \leq |\mathsf{BL}(\mathcal{S})| \leq 2^{2^{\aleph_0}}$, where the second inequality was proved in [Ohk89]. □

Lemma 2.8. *Let T be as in Definition 2.1, and X and Y objects of T. Then $\langle X\rangle \leq \langle Y\rangle$ if and only if $L_X = L_X L_Y = L_Y L_X$ and in this case the following diagram commutes (also with BL replaced by DL or BA).*

$$\begin{array}{ccc} & \mathsf{BL}(\mathsf{T}) & \\ {}^{L_Y}\swarrow & & \searrow^{L_X} \\ \mathsf{BL}(\mathcal{L}_Y) & \xrightarrow{\;L_X\;} & \mathsf{BL}(\mathcal{L}_X) \end{array}$$

Proof. The first equivalence is straightforward; it follows from [Rav84, Prop. 1.22] and the observation that $\langle X\rangle \leq \langle Y\rangle$ precisely when all L_X-locals are L_Y-locals. The last remark follows from Proposition 2.6. □

Here we mention one object-level application of the Bousfield lattice calculations of Sections 5 and 6. Call an object X *square-zero* if it is nonzero, but $X \wedge X = 0$. For example, $I \wedge I = 0$ in $\mathcal{S}$.

Proposition 2.9. There are no square-zero objects in the harmonic, $E(n)$-, $K(n)$-, $H\mathbb{F}_p$-, or I-local categories.

Proof. In Corollary 2.8 of [Wol13], we show that in a well generated tensor triangulated category, there are no square-zero objects if and only if $\mathsf{BA} = \mathsf{DL} = \mathsf{BL}$. The claim then follows from Corollaries 5.4 and 5.10 and Propositions 6.2 and 6.6. □

2.4. **Background on spectra.** We quickly review some relevant background on the stable homotopy category. See [Rav92, Hov95, MS95, Rav93] for more details. Fix a prime p and let $\mathcal{S}$ denote the p-local stable homotopy category of spectra. Let S^0 denote the sphere spectrum. The *finite* spectra $\mathcal{F}$ are the compact objects of $\mathcal{S}$, and $\mathcal{F} = \mathsf{th}(S^0)$. The structure of $\mathcal{F}$ is determined by the Morava K-theories $K(i)$. For each $i \geq 0$, $K(i)$ is a skew field object in $\mathcal{S}$, such that $K(i) \wedge K(j) = 0$ when $i \neq j$. If X is a finite spectrum and $K(j) \wedge X = 0$, then $K(j-1) \wedge X = 0$. We say a finite spectrum X is *type* n if n is the smallest integer such that $K(n) \wedge X \neq 0$. Define $\mathcal{C}_n = \langle K(n-1) \rangle \cap \mathcal{F}$. Then every thick subcategory of $\mathcal{F}$ is $\mathcal{C}_n$ for some n. It follows that any two spectra of type n generate the same thick subcategory, and hence Bousfield class; let $F(n)$ denote a generic type n spectrum.

Given a type n spectrum X, there is a v_n self-map $f : \Sigma^d X \to X$ with $[S^0, K(n) \wedge f]_i$ an isomorphism for all i and $[S^0, K(m) \wedge f]_j = 0$ for all j and $m \neq n$. We define $f^{-1}X$ to be the telescope, i.e. sequential colimit, i.e. homotopy colimit, of the diagram $X \to \Sigma^{-d} X \to \cdots$. By the periodicity theorem, any choice of v_n self-map f yields an isomorphic telescope. The telescopes of different type n spectra are Bousfield equivalent; denote this class by $\langle T(n) \rangle$.

As mentioned above, localization away from a finite spectrum $F(n+1)$ exists, and is smashing. This localization functor is denoted L_n^f, and is the same as homological localization at $T(0) \vee \cdots \vee T(n)$.

Let $E(n)$ denote the Johnson-Wilson spectrum; this is a ring spectrum with $\langle E(n) \rangle = \langle K(0) \vee \cdots \vee K(n) \rangle$. Define $L_n : \mathcal{S} \to \mathcal{S}$ to be homological localization at $E(n)$. A deep theorem of Ravenel [Rav92, Thm. 7.5.6] shows that L_n is smashing for all n. The L_n^f and L_n are the only known smashing localization functors on $\mathcal{S}$.

The L_n^f-acyclics are $\mathsf{loc}(F(n+1)) = \mathsf{loc}(\mathcal{C}_{n+1}) = \mathsf{loc}(\langle K(n) \rangle \cap \mathcal{F}) = \mathsf{loc}(\langle E(n) \rangle \cap \mathcal{F})$. The L_n-acyclics are $\langle E(n) \rangle = \mathsf{loc}(\langle E(n) \rangle)$. Thus every L_n^f-acyclic is L_n-acyclic, and we have $\langle K(0) \vee \cdots \vee K(n) \rangle \leq \langle T(0) \vee \cdots \vee T(n) \rangle$ for all n. It follows that there is a natural map $L_n^f \to L_n$.

For convenience later, we collect some calculations in $\mathcal{S}$.

Lemma 2.10. *In* $\mathsf{BL}(\mathcal{S})$ *we have the following.*

(1) $\langle F(m) \rangle \leq \langle F(n) \rangle$ *if and only if* $m \geq n$. *For all* n *and* m, $\langle F(m) \wedge F(n) \rangle \neq \langle 0 \rangle$. *Furthermore,* $\langle F(n) \wedge F(n) \rangle = \langle F(n) \rangle$ *for all* n.
(2) $\langle F(m) \wedge T(n) \rangle = \langle 0 \rangle$ *when* $m > n$, *and* $\langle F(m) \wedge T(n) \rangle = \langle T(n) \rangle$ *when* $m \leq n$.
(3) $\langle T(m) \wedge T(n) \rangle = \langle 0 \rangle$ *when* $m \neq n$, *and* $\langle T(n) \wedge T(n) \rangle = \langle T(n) \rangle$.
(4) $\langle F(m) \wedge K(n) \rangle = \langle 0 \rangle$ *when* $m > n$, *and* $\langle K(n) \rangle = \langle F(m) \wedge K(n) \rangle \leq \langle F(m) \rangle$ *when* $m \leq n$.
(5) $\langle T(m) \wedge K(n) \rangle = \langle 0 \rangle$ *when* $m \neq n$, *and* $\langle K(n) \rangle = \langle T(n) \wedge K(n) \rangle \leq \langle T(n) \rangle$.
(6) $\langle K(m) \wedge K(n) \rangle = \langle 0 \rangle$ *when* $m \neq n$, *and* $\langle K(n) \wedge K(n) \rangle = \langle K(n) \rangle$.

Proof. Part (1) is Theorem 14 of [HS98], along with the observation that $\langle F(n) \rangle$ is complemented by $\langle L_{n-1}^f S^0 \rangle$ and hence is in DL.

Part (2) is in [Rav93, 2.8(i)], [MS95, 6.2], and [HP99, Sect. 5]. Part (3) is also in [HP99, Sect. 5].

Part (4) follows from the definition of type m spectra. Since each $K(i)$ is a skew field object, $F(m) \wedge K(i) \neq \langle 0 \rangle$ implies this $K(i)$-module object $F(m) \wedge K(i)$ is a wedge of suspensions of $K(i)$.

From the periodicity theorem, $T(m)$ has nonzero $K(m)$ homology, so $T(m) \wedge K(m) \neq 0$. The rest of Part (5) is in [Rav92, Prop. A.2.13]. Finally, Part (6) is well known. □

3. Local versions of the telescope conjecture

In this section, let $L = L_Z : \mathcal{S} \to \mathcal{S}$ be a localization functor for some $Z \in \mathcal{S}$, and let $\mathcal{L} = \mathcal{L}_Z$ denote the category of L-locals. First we state the various versions of the original telescope conjecture on $\mathcal{S}$.

Definition 3.1. Fix an integer $n \geq 0$. On $\mathcal{S}$, we have the following versions of the telescope conjecture.

$\mathsf{TC1_n}$:	$\langle T(n)\rangle = \langle K(n)\rangle$.
$\mathsf{TC2_n}$:	$L_n^f X \xrightarrow{\sim} L_n X$ for all X.
$\mathsf{TC3_n}$:	If X is type n and f is a v_n self-map, then $L_n X \cong f^{-1}X$.
GSC :	Every smashing localization is generated by a set of compact objects.
SDGSC :	Every smashing localization is generated by a set of strongly dualizable objects.

Theorem 3.2. On the category $\mathcal{S}$ we have $\mathsf{TC1_n} \Leftrightarrow \mathsf{TC3_n}$.
Also, $\mathsf{TC2_n}$ holds if and only if $\mathsf{TC1_i}$ holds for all $i \leq n$.
Given $\mathsf{TC2_{n-1}}$ and $\mathsf{TC1_n}$, then $\mathsf{TC2_n}$ holds.
Furthermore, $\mathsf{GSC} \Leftrightarrow \mathsf{SDGSC}$, and this implies $\mathsf{TC2_n}$ for all n.

Remark 3.3. Note that if we quantify over all n, the first three versions of the telescope conjecture are equivalent. That is,

$$\mathsf{TC1_n} \text{ for all } n \Leftrightarrow \mathsf{TC2_n} \text{ for all } n \Leftrightarrow \mathsf{TC3_n} \text{ for all } n.$$

Remark 3.4. $\mathsf{TC2_n}$ holds if and only if $L_n^f S^0 \xrightarrow{\sim} L_n S^0$. Indeed, since both L_n^f and L_n are smashing, the subcategory of objects W such that $L_n^f W \xrightarrow{\sim} L_n W$ is localizing. Thus if it contains S^0, it contains $\mathsf{loc}(S^0) = \mathcal{S}$.

Proof. First we show the equivalence of $\mathsf{TC1_n}$ and $\mathsf{TC3_n}$. This is also sketched in [MRS01, 1.13]. For any type n spectrum Y, $\mathsf{th}(Y) = \mathsf{th}(F(n))$ and so we have $\mathsf{th}(L_n Y) = \mathsf{th}(L_n F(n))$, and $\langle L_n Y\rangle = \langle L_n F(n)\rangle$. A construction in [Rav92, 8.3] gives a type n spectrum Y with $L_n Y \in \mathsf{th}(K(n))$. Thus $\langle L_n Y\rangle \leq \langle K(n)\rangle$, and $0 \neq \langle L_n F(n)\rangle = \langle K(n)\rangle$. Suppose $\mathsf{TC3_n}$ holds. Then $\langle L_n Y\rangle = \langle f^{-1}Y\rangle = \langle T(n)\rangle$, and so $\langle T(n)\rangle = \langle K(n)\rangle$.

If X is type n and f is a v_n self-map, then [MS95, Prop. 3.2] implies that $L_n^f X \cong L_{T(n)} X \cong f^{-1}X$. Thus assuming $\mathsf{TC1_n}$, we have $L_{K(n)} X \cong f^{-1}X$, and to prove $\mathsf{TC3_n}$ it suffices to show that $L_n X \cong L_{K(n)} X$. This is known (see e.g. [Hov95]), but we will give a proof that extends well to the localized setting. Since $\langle K(n)\rangle \leq \langle E(n)\rangle$, localization at $K(n)$ gives a map $L_n X \to L_{K(n)} X$. It suffices to show that this is an L_n-equivalence. The fiber is $K(n)$ acyclic, so $L_n X \wedge K(n) \to L_{K(n)} X \wedge K(n)$ is an isomorphism. Consider $i < n$. The triangle $C_n X \wedge K(i) \to X \wedge K(i) \to L_n X \wedge K(i)$ shows that $L_n X \wedge K(i)$ is zero, because X is type n and $C_n X$ is $K(i)$ acyclic. Lemma 3.3.1 in [HPS97] states that $LW = LS^0 \wedge W$ for any localization

L and strongly dualizable W. Since every finite spectrum is strongly dualizable, $L_{K(n)}X \wedge K(i) = L_{K(n)}S^0 \wedge X \wedge K(i) = 0$. Thus $L_nX \wedge K(i) \to L_{K(n)}X \wedge K(i)$ is an isomorphism for all $i \leq n$, and hence $L_nX \to L_{K(n)}X$ is an L_n-equivalence.

For the second statement, note that $\mathsf{TC2_n}$ is equivalent to the statement $\langle T(0) \vee \cdots \vee T(n)\rangle = \langle K(0) \vee \cdots \vee K(n)\rangle$. Smashing this with $\langle T(i)\rangle$, for $0 \leq i \leq n$, and using Lemma 2.10, yields $\mathsf{TC1_i}$ for each i. The third statement is also clear from this observation.

Finally, $\mathsf{GSC} \Leftrightarrow \mathsf{SDGSC}$ because objects in $\mathcal{S}$ are compact if and only if they are strongly dualizable. Given GSC, consider L_n. The GSC would imply that the L_n-acyclics are $\mathsf{loc}(\langle E(n)\rangle \cap \mathcal{F})$. As observed earlier, this is the same as $\mathsf{loc}(F(n+1))$, which are the L_n^f-acyclics. Therefore we would have $L_n^f \cong L_n$. □

The SDGSC is new, and we will discuss it first. As mentioned in the above proof, in $\mathcal{S}$ compactness is equivalent to strong dualizability. It is well known that localization away from a set of compact objects is smashing. The GSC is precisely the statement that the converse holds. However, as we will show next, one only needs strong dualizability to generate a smashing localization functor. We will state our result in slightly more general terms.

Theorem 3.5. Let T be a well generated tensor triangulated category such that $\mathsf{loc}(\mathbb{1}) = \mathsf{T}$, as in Definition 2.1. Let $A = \{B_\alpha\}$ be a (possibly infinite) set of strongly dualizable objects. Then there exists a smashing localization functor $L : \mathsf{T} \to \mathsf{T}$ with Ker $L = \mathsf{loc}(A)$.

Proof. Let $E = \vee_\alpha B_\alpha$ and note that $\mathsf{loc}(E) = \mathsf{loc}(A)$. The category T is well generated by hypothesis. The localizing subcategory $\mathsf{S} = \mathsf{loc}(E)$ is also well generated, by [IK13, Rmk. 2.2], and is tensor-closed by Lemma 2.5.

By [IK13, Prop. 2.1] there exists a localization functor $L : \mathsf{T} \to \mathsf{T}$ with Ker $L = \mathsf{S}$. We will show that L is a smashing localization.

First we claim that the L-locals are tensor-closed. For any $Y \in \mathsf{T}$, we have the following.

$$\begin{aligned} Y \text{ is } L\text{-local} \quad &\Leftrightarrow [W, Y]_n = 0 \text{ for all } W \in \mathsf{S} \text{ and all } n \in \mathbb{Z} \\ &\Leftrightarrow [E, Y]_n = \textstyle\prod [B_\alpha, Y]_n = 0 \text{ for all } n \in \mathbb{Z} \\ &\Leftrightarrow [B_\alpha, Y]_n = 0 \text{ for all } \alpha \text{ and } n \in \mathbb{Z} \\ &\Leftrightarrow DB_\alpha \wedge Y = 0 \text{ for all } \alpha \end{aligned}$$

The second equivalence follows from the fact that $\{X \mid [X, Y]_n = 0 \text{ for all } n \in \mathbb{Z}\}$ is a localizing subcategory containing E, hence all of S. The final equivalence uses the fact that the B_α are strongly dualizable.

Now suppose Y is L-local and X is arbitrary. Then $DB_\alpha \wedge Y = 0$ for all α, so $DB_\alpha \wedge Y \wedge X = 0$ for all α, and thus $Y \wedge X$ is L-local. This shows that the L-locals are tensor-closed.

Consider the localization triangle $C\mathbb{1} \to \mathbb{1} \to L\mathbb{1}$, where $L\mathbb{1}$ is L-local and $C\mathbb{1} \in \mathsf{S}$. For arbitrary $X \in \mathsf{T}$, tensoring gives an exact triangle

$$C\mathbb{1} \wedge X \to X \to L\mathbb{1} \wedge X.$$

The object $L\mathbb{1} \wedge X$ is L-local, since the locals are tensor-closed. Likewise, $C\mathbb{1} \wedge X \in \mathsf{S}$, since S is tensor-closed and so $L(C\mathbb{1} \wedge X) = 0$. Therefore $X \to L\mathbb{1} \wedge X$ is an L-equivalence from X to an L-local object, and it follows that $LX \cong L\mathbb{1} \wedge X$. This shows that L is a smashing localization. □

In the stable homotopy category, and more generally whenever $\mathbb{1} \in \mathsf{T}$ is compact, this gives nothing new; by [HPS97, Thm. 2.1.3(d)] compact and strongly dualizable are equivalent. Consider, however, the harmonic category, which has no nonzero compact objects. In Section 4 we classify all smashing localizations on the harmonic category; they are indexed by $\mathbb{N}$. Thus the GSC fails in the harmonic category, but as we show in Theorem 4.4, the SDGSC holds. In fact, in the following sections we will give several examples of categories where the GSC fails but the SDGSC holds.

On the other hand, we don't expect the SDGSC to hold in complete generality, since Keller's counterexample [Kel94] to the GSC is also a counterexample to the SDGSC; in the derived category of a ring R, the unit R is compact and strongly dualizable, so the GSC and SDGSC are equivalent.

The GSC and SDGSC make sense in any localized category, but $\mathsf{TC1_n}$, $\mathsf{TC2_n}$, and $\mathsf{TC3_n}$ may not, since $T(n)$ and $K(n)$ may not be objects in $\mathcal{L}$. Instead we make the following definitions.

Definition 3.6. Let $L : \mathcal{S} \to \mathcal{S}$ be a localization, and $\mathcal{L}$ the category of L-locals.

(1) Let $l_n^f : \mathcal{L} \to \mathcal{L}$ denote localization at $\langle LT(0) \vee LT(1) \vee \cdots \vee LT(n)\rangle$.
(2) Let $l_n : \mathcal{L} \to \mathcal{L}$ denote localization at $\langle LK(0) \vee LK(1) \vee \cdots \vee LK(n)\rangle$.

Before stating and proving a local version of Theorem 3.2, we establish some results about l_n^f and l_n. First we make an observation about calculations in $\mathsf{BL}(\mathcal{L})$.

Lemma 3.7. *All the calculations in Lemma 2.10 are valid in* $\mathsf{BL}(\mathcal{L})$ *if we replace* $F(n)$, $T(n)$, *and* $K(n)$ *with* $LF(n)$, $LT(n)$, *and* $LK(n)$.

Proof. This follows from Theorem 2.3 and the statements in Lemma 2.10. □

Proposition 3.8. l_n^f is localization away from $LF(n+1)$, and hence is smashing.

Proof. By Theorem 3.5 we know that there is some smashing localization functor $l : \mathcal{L} \to \mathcal{L}$ that is localization away from $LF(n+1)$; we wish to show $l = l_n^f$. Let $\mathbb{1} = LS^0$ for simplicity of notation, and let c denote the colocalization corresponding to l. We claim that the l-acyclics are precisely $\mathsf{loc}(c\mathbb{1})$. Clearly $c\mathbb{1}$ is l-acyclic, and these are a localizing subcategory, so $\mathsf{loc}(c\mathbb{1}) \subseteq \{l\text{-acyclics}\}$. On the other hand, suppose W is l-acyclic. Because l is smashing, $LW = W \wedge l\mathbb{1} = 0$, so $W = W \wedge c\mathbb{1}$. Then since $W \in \mathsf{loc}(\mathbb{1}) = \mathcal{L}$, we have $W = W \wedge c\mathbb{1} \in \mathsf{loc}(\mathbb{1} \wedge c\mathbb{1}) = \mathsf{loc}(c\mathbb{1})$, proving the claim.

By definition, the l-acyclics are also given by $\mathsf{loc}(LF(n+1))$. Therefore $\langle LF(n+1)\rangle = \langle c\mathbb{1}\rangle$.

The class $\langle F(n+1)\rangle$ is complemented by $\langle T(0) \vee \cdots \vee T(n)\rangle$ in $\mathsf{BL}(\mathcal{S})$ [HP99, Sect. 5], so $\langle LF(n+1)\rangle$ is complemented by $\langle LT(0) \vee \cdots \vee LT(n)\rangle$ in $\mathsf{BL}(\mathcal{L})$. At the same time, $\langle c\mathbb{1}\rangle$ is complemented by $\langle l\mathbb{1}\rangle$, and complements are unique. We conclude that $\{l\text{-acylics}\} = \langle l\mathbb{1}\rangle = \langle LT(0) \vee \cdots \vee LT(n)\rangle$. Since l and l_n^f are two localizations on $\mathcal{L}$ with the same acyclics, they are equal. □

Lemma 3.9. *If L is smashing, then $l_n^f = LL_n^f = L_n^f L$ and $l_n = LL_n = L_nL$, and both are smashing.*

Proof. Smashing localization functors always commute, and compose to give a smashing localization. The functor $LL_n : \mathcal{S} \to \mathcal{S}$, sending $X \mapsto L(L_nS^0 \wedge X) = LS^0 \wedge L_nS^0 \wedge X$ is a smashing localization. Since LL_n-locals are L-local, it also gives a smashing localization on $\mathcal{L}$. The acyclics of this functor are $\langle LL_nS^0\rangle$ in

$\mathsf{BL}(\mathcal{L})$, which is $\langle LK(0) \vee \cdots \vee LK(n)\rangle$. Thus LL_n and l_n are localizations on $\mathcal{L}$ with the same acyclics, hence isomorphic. The same proof works for $l_n^f = LL_n^f$. $\square$

In $\mathcal{S}$, for a type n finite spectrum X with a v_n map $f : \Sigma^d X \to X$ and telescope $f^{-1}X$, it is known [MS95, Prop. 3.2] that $L_n^f X \cong L_{T(n)}X \cong f^{-1}X$. The following proposition shows that the local version of this result holds as well.

Lemma 3.10. *Let $L : \mathcal{S} \to \mathcal{S}$ be a localization, and l_n^f, X and $f^{-1}X$ as above. Then*

$$l_n^f(LX) \cong L_{LT(n)}(LX) \cong L(f^{-1}X).$$

Proof. The proof parallels the [MS95] result, one must only check that everything works when localized. If $L_{LT(n)}(LX) \cong L(f^{-1}X)$ holds for a single type n spectrum, then it holds for all type n spectra. So without loss of generality, we can choose X to be a type n spectrum that is a ring object in $\mathcal{S}$. Then for any v_n self-map f, the telescope $f^{-1}X$ is also a ring object [MS95, Lemma 2.2]. By Lemma 2.5, $L(f^{-1}X)$ is a ring object in $\mathcal{L}$, and hence is local with respect to itself.

Lemma 2.2 in [MS95] shows that $X \wedge f^{-1}X \cong f^{-1}X \wedge f^{-1}X$ in $\mathcal{S}$, so $LX \wedge_{\mathcal{L}} L(f^{-1}X) \cong L(f^{-1}X) \wedge_{\mathcal{L}} L(f^{-1}X)$ in $\mathcal{L}$ and the canonical map $LX \to L(f^{-1}X)$ is an $L(f^{-1}X)$-equivalence. It follows that $L_{LT(n)}(LX) \cong L(f^{-1}X)$.

Since $\langle LT(n)\rangle \leq \langle LT(0) \vee \cdots \vee LT(n)\rangle$, $L(f^{-1}X)$ is l_n^f-local. One then uses Lemma 3.7 to see that $LX \to L(f^{-1}X)$ is a l_n^f-equivalence, and so $l_n^f(LX) = l_n^f(L(f^{-1}X)) = L(f^{-1}X)$. $\square$

Definition 3.11. Let $L : \mathcal{S} \to \mathcal{S}$ be a localization, and consider the category $\mathcal{L}$ of locals. Fix an $n \geq 0$. We have the following versions of the telescope conjecture on $\mathcal{L}$.

- $\mathsf{LTC1_n}$: $\langle LT(n)\rangle = \langle LK(n)\rangle$.
- $\mathsf{LTC2_n}$: $l_n^f X \xrightarrow{\sim} l_n X$ for all X.
- $\mathsf{LTC3_n}$: If $X \in \mathcal{S}$ is type n and f is a v_n self-map, then $l_n(LX) \cong L(f^{-1}X)$.
- GSC : Every smashing localization is generated by a set of compact objects.
- SDGSC : Every smashing localization is generated by a set of strongly dualizable objects.

Theorem 3.12. On the category $\mathcal{L}$ we have $\mathsf{LTC1_n} \Rightarrow \mathsf{LTC3_n}$.
Also, $\mathsf{LTC2_n}$ holds if and only if $\mathsf{LTC1_i}$ holds for all $i \leq n$.
Given $\mathsf{LTC2_{n-1}}$ and $\mathsf{LTC1_n}$, then $\mathsf{LTC2_n}$ holds.

Proof. Note that $\mathsf{LTC2_n}$ is equivalent to the statement $\langle LT(0) \vee \cdots \vee LT(n)\rangle = \langle LK(0) \vee \cdots \vee LK(n)\rangle$, so the last two statements are clear. We will show that $\mathsf{LTC1_n} \Rightarrow \mathsf{LTC3_n}$, by mimicking the proof in Theorem 3.2.

If X is type n and f is a v_n self-map, Lemma 3.10 shows that $l_n^f(LX) \cong L_{LT(n)}(LX) \cong L(f^{-1}X)$. Then $\mathsf{LTC1_n}$ implies $L_{LK(n)}LX \cong L(f^{-1}X)$. So it suffices to show that $l_n(LX) = L_{LK(n)}LX$. We must show that the map $L_{LK(n)} : l_n(LX) \to L_{LK(n)}LX$ is an l_n-equivalence. The same reasoning as in Theorem 3.2, along with the computations of Lemma 3.7 and some definition unwinding, gives us that $l_n(LX) \wedge LK(i) \to L_{LK(n)}LX \wedge LK(i)$ is an equivalence for all $i \leq n$; we only need to notice that Lemma 3.3.1 in [HPS97] applies to strongly dualizable objects, and LX is strongly dualizable. $\square$

Theorem 3.13. Furthermore, if $L : \mathcal{S} \to \mathcal{S}$ is a smashing localization, then on the category $\mathcal{L}$ of locals we have $\mathsf{LTC1_n} \Leftarrow \mathsf{LTC3_n}$, and

$$\mathsf{GSC} \Leftrightarrow \mathsf{SDGSC} \Rightarrow \mathsf{LTC2_n} \text{ for all } n.$$

Remark 3.14. In this case, $\mathsf{LTC2_n}$ is equivalent to $l_n^f(LS^0) \xrightarrow{\sim} l_n(LS^0)$, since by Lemma 3.9 both l_n^f and l_n are smashing, so the argument in Remark 3.4 applies.

Proof. By [HPS97, Thm. 2.1.3(d)], the compact objects and strongly dualizable objects in $\mathcal{L}$ coincide. Thus $\mathsf{GSC} \Leftrightarrow \mathsf{SDGSC}$, and this implies $\mathsf{LTC2_n}$ just as in Theorem 3.2.

Suppose X has type n and $\mathsf{LTC3_n}$ holds. As in the proof of Theorem 3.2, $\langle L_nF(n)\rangle = \langle K(n)\rangle$ in $\mathsf{BL}(\mathcal{S})$, so $\langle LL_nLF(n)\rangle = \langle LK(n)\rangle$ in $\mathsf{BL}(\mathcal{L})$. By Lemma 3.9, we have $\langle l_nLF(n)\rangle = \langle LK(n)\rangle$. Then $\mathsf{LTC3_n}$ implies that $\langle LT(n)\rangle = \langle L(f^{-1}X\rangle = \langle l_n(LX)\rangle = \langle l_nLF(n)\rangle$, so $\mathsf{LTC1_n}$ holds. □

Question 3.15. Is $l_n : \mathcal{L} \to \mathcal{L}$ always a smashing localization?

This is the case in all the local categories investigated in this paper, whether or not $L : \mathcal{S} \to \mathcal{S}$ is a smashing localization. If one could show l_n is always smashing, then most likely on $\mathcal{L}$ one would have $\mathsf{SDGSC} \implies \mathsf{LTC2_n} \forall n$.

We would of course like to know if and when information on localized telescope conjectures can help with those in the original category $\mathcal{S}$, where all versions remain open.

Proposition 3.16. Let $L : \mathcal{S} \to \mathcal{S}$ be a localization, with localized category $\mathcal{L}$.

(1) If $\mathsf{TC1_n}$ holds on $\mathcal{S}$, then $\mathsf{LTC1_n}$ holds on $\mathcal{L}$.
(2) If $\mathsf{TC2_n}$ holds on $\mathcal{S}$, then $\mathsf{LTC2_n}$ holds on $\mathcal{L}$.
(3) If $\mathsf{TC3_n}$ holds on $\mathcal{S}$, then $\mathsf{LTC3_n}$ holds on $\mathcal{L}$.

Furthermore, if L is a smashing localization then we have the following.

(4) If GSC holds on $\mathcal{S}$, then GSC holds on $\mathcal{L}$.
(5) If SDGSC holds on $\mathcal{S}$, then SDGSC holds on $\mathcal{L}$.

Proof. Part (1) follows immediately from Proposition 2.6. So does Part (2), since $\mathsf{TC2_n}$ is equivalent to the statement $\langle T(0) \vee \cdots \vee T(n)\rangle = \langle K(0) \vee \cdots \vee K(n)\rangle$ and similarly for $\mathsf{LTC2_n}$. From this and Theorems 3.2 and 3.12 we have $\mathsf{TC3_n} \Leftrightarrow \mathsf{TC1_n} \Rightarrow \mathsf{LTC1_n} \Rightarrow \mathsf{LTC3_n}$.

Now suppose L is smashing, and the GSC holds on $\mathcal{S}$. Let $l : \mathcal{L} \to \mathcal{L}$ be a smashing localization. Thus l is defined by $l(LY) = l(LS^0) \wedge_{\mathcal{L}} LY = lS^0 \wedge LS^0 \wedge Y$. We can therefore extend l to be a smashing localization on all of $\mathcal{S}$, with $X \mapsto lS^0 \wedge LS^0 \wedge X = lLS^0 \wedge X$. Since the GSC holds on $\mathcal{S}$ by assumption, the acyclics of this functor are $\langle lLS^0\rangle = \mathsf{loc}(A)$, for some set of compact objects A in $\mathcal{S}$. Here $\langle lLS^0\rangle$ refers to the Bousfield class in $\mathsf{BL}(\mathcal{S})$.

We must show that $\langle lLS^0\rangle$ in $\mathsf{BL}(\mathcal{L})$ is generated by a set of objects that are compact in $\mathcal{L}$. Note that $\langle lLS^0\rangle$ in $\mathsf{BL}(\mathcal{L})$ is $\{LW \mid LW \wedge_{\mathcal{L}} lLS^0 = 0\} = \{LW \mid LW \wedge_{\mathcal{S}} lLS^0 = 0\} = \langle lLS^0\rangle \cap \mathcal{L}$, where the latter $\langle lLS^0\rangle$ is in $\mathsf{BL}(\mathcal{S})$. Therefore $\langle lLS^0\rangle$ in $\mathsf{BL}(\mathcal{L})$ is $\mathsf{loc}(A) \cap \mathcal{L}$. We claim that this is $\mathsf{loc}(L(A))$. Since L sends compacts to compacts, this will show that l is generated by a set of compacts.

If $X \in \mathsf{loc}(A)$, then $LX \in \mathsf{loc}(L(A))$. If $X \in \mathcal{L}$ in addition, then $X \cong LX \in \mathsf{loc}(L(A))$. For the other inclusion, note that the intersection of two localizing subcategories is a localizing subcategory, and $\mathcal{L}$ is a localizing subcategory of $\mathcal{S}$

because L is smashing. For $Y \in A$, LY is in $\mathcal{L}$, and $LY = LS^0 \wedge Y \in \mathsf{loc}(Y) \subseteq \mathsf{loc}(A)$. Therefore $L(A) \subseteq \mathsf{loc}(A) \cap \mathcal{L}$, and $\mathsf{loc}(L(A)) = \mathsf{loc}(A) \cap \mathcal{L}$.

Part (5) follows immediately, since if L is smashing then $\mathsf{GSC} \Leftrightarrow \mathsf{SDGSC}$ in both $\mathcal{S}$ and $\mathcal{L}$. □

Balmer and Favi [BF11, Prop. 4.4] have also recently proved Part (4) in the slightly more general setting of a smashing localization on a unital algebraic stable homotopy category; the above proof would apply there as well. One would like to prove Part (5) without the assumption that L is smashing, but it's not clear if this is possible.

Letting $L = L_Z$, for $Z = \bigvee_{i\geq 0} K(i)$, $E(n)$, $K(n)$, BP, $H\mathbb{F}_p$, or I provides interesting examples of categories $\mathcal{L}$ on which to investigate these telescope conjectures. Furthermore, $\mathsf{LTC1_n}$ suggests the relevance of Bousfield lattices to understanding these questions. In the remaining sections, we focus on specific localized categories.

4. The Harmonic category

Let $Q = \bigvee_{i\geq 0} K(i)$, and $L = L_Q : \mathcal{S} \to \mathcal{S}$, and consider the harmonic category $\mathcal{H}$ of L-locals. Harmonic localization is not smashing. An object is called *harmonic* if it is L-local, and *dissonant* if it is L-acyclic. For example, finite spectra, suspension spectra, finite torsion spectra, and BP are known to be harmonic [Hov95, Rav84]. On the other hand, I and $H\mathbb{F}_p$ are dissonant.

In order to answer the telescope conjectures in $\mathcal{H}$, we will first calculate the Bousfield lattice of $\mathcal{H}$. In this section all smash products are in $\mathcal{H}$, unless otherwise noted. Given any set P, let 2^P denote the power set of P.

Definition 4.1. Given $X \in \mathcal{H}$, define the *support* of X to be

$$\mathsf{supp}(X) = \{i \mid X \wedge K(i) \neq 0\} \subseteq \mathbb{N}.$$

The following result and proof was pointed out to us by Jon Beardsley.

Proposition 4.2. The Bousfield lattice of $\mathcal{H}$ is $2^{\mathbb{N}}$.

Proof. Each $K(n)$ is a ring object, hence $K(n)$-local by Lemma 2.5. Because $\langle K(n)\rangle \leq \langle Q\rangle$, $K(n)$-locals are harmonic, thus each $K(n)$ is harmonic. The argument hinges on the fact that $K(n)$ is a skew field object in $\mathcal{H}$: for $X = LX$ in $\mathcal{H}$, if $X \wedge K(n) \neq 0$ then $X \wedge K(n) = L(X \wedge_{\mathcal{S}} K(n))$ so $X \wedge_{\mathcal{S}} K(n) \neq 0$, and $X \wedge_{\mathcal{S}} K(n)$ must be a nonempty wedge of suspensions of $K(n)$'s. Thus

$$X \wedge K(n) = L(X \wedge_{\mathcal{S}} K(n)) = L(\vee \Sigma^i K(n)) = L(\vee \Sigma^i LK(n)) = \coprod_{\mathcal{L}} \Sigma^i LK(n) = \coprod_{\mathcal{L}} \Sigma^i K(n).$$

It follows that $LX \wedge K(n) = 0$ if and only if $LX \wedge_{\mathcal{S}} K(n) = 0$. Furthermore, if $LX \wedge K(n) \neq 0$, then $\langle LX \wedge K(n)\rangle = \langle K(n)\rangle$, where these are Bousfield classes in $\mathsf{BL}(\mathcal{H})$.

By the definition of L, for any $W \in \mathcal{S}$, if $W \wedge_{\mathcal{S}} K(n) = 0$ for all n, then $LW = 0$. Combining this with the above observation, we get that a local object $W = LW$ has $W \wedge K(n) = 0$ in $\mathcal{H}$ for all n if and only if $W = 0$.

Therefore, for any $X, Y \in \mathcal{H}$, we have

$$Y \wedge X = 0 \Leftrightarrow Y \wedge X \wedge K(n) = 0 \ \forall n \Leftrightarrow Y \wedge K(n) = 0 \text{ for all } n \in \mathsf{supp}(X).$$

We conclude that there is a lattice isomorphism $F : \mathsf{BL}(\mathcal{H}) \to 2^{\mathbb{N}}$, given by the following.

$$\langle X \rangle = \bigvee_{\mathsf{supp}(X)} \langle K(i) \rangle \mapsto \mathsf{supp}(X),$$

$$N \subseteq \mathbb{N} \mapsto \bigvee_{i \in N} \langle K(i) \rangle.$$

□

Theorem 4.3. On $\mathcal{H}$, for all $n \geq 0$ we have that $\mathsf{LTC1_n}, \mathsf{LTC2_n}$, and $\mathsf{LTC3_n}$ hold.

Proof. By Lemmas 2.10 and 3.7, $LT(n)$ and $LK(n) = K(n)$ have the same support. The above theorem then implies that $\langle LT(n) \rangle = \langle LK(n) \rangle$. Thus $\mathsf{LTC1_n}$ holds for all n, and the claim follows from Theorem 3.12. □

Next, we classify all smashing localizations on $\mathcal{H}$, and show that the GSC fails but the SDGSC holds. The proof is based on that of [HS99, Thm. 6.14], which classifies smashing localizations in the $E(n)$-local category.

Theorem 4.4. If $L' : \mathcal{H} \to \mathcal{H}$ is a smashing localization functor, then $L' = l_n^f$ for some $n \geq 0$, or $L' = 0$ or $L' = id$. Therefore the GSC fails but the SDGSC holds on $\mathcal{H}$.

Proof. Let $L' : \mathcal{H} \to \mathcal{H}$ be a smashing localization functor, and let $\mathbb{1} = LS^0$ be the unit in $\mathcal{H}$. The acyclics of L' are given by $\langle L'\mathbb{1} \rangle$. From Proposition 4.2, $\langle L'\mathbb{1} \rangle$ is equal to the wedge of $\langle K(i) \rangle$ for all $i \in \mathsf{supp}(L'\mathbb{1})$. If $\mathsf{supp}(L'\mathbb{1}) = \emptyset$ then $\langle L'\mathbb{1} \rangle = \langle 0 \rangle$ and $L' = 0$.

Assume now that $\mathsf{supp}(L'\mathbb{1})$ is not empty, and take $j \in \mathsf{supp}(L'\mathbb{1})$. We will show that $\langle L'\mathbb{1} \rangle \geq \langle K(0) \vee \cdots \vee K(j) \rangle$. It follows that either $\langle L'\mathbb{1} \rangle = \bigvee_{i \geq 0} \langle K(i) \rangle = \langle \mathbb{1} \rangle$ and $L' = id$, or $L' = l_n = l_n^f$ for $n = max(\mathsf{supp}(L'\mathbb{1}))$.

Since $\langle K(j) \rangle \leq \langle L'\mathbb{1} \rangle$, from Lemma 2.8 we have $L_{K(j)}L' = L'L_{K(j)} = L_{K(j)}$. Therefore $\langle L_{K(j)}\mathbb{1} \rangle = \langle L'\mathbb{1} \wedge L_{K(j)}\mathbb{1} \rangle \leq \langle L'\mathbb{1} \rangle$. Proposition 5.3 of [HS99] shows that in $\mathcal{S}$, $L_{K(j)}S^0 \wedge_{\mathcal{S}} K(i)$ is nonzero for $0 \leq i \leq j$ and zero for $i > j$. Note that $L_{K(j)}S^0 = L_{K(j)}LS^0 = L_{K(j)}\mathbb{1}$, and as remarked in the proof of Proposition 4.2, $LX \wedge K(i) = 0$ if and only if $LX \wedge_{\mathcal{S}} K(i) = 0$. Therefore in $\mathsf{BL}(\mathcal{H})$ we have $\langle L_{K(j)}\mathbb{1} \rangle = \langle K(0) \vee \cdots \vee K(j) \rangle$, and so $\langle L'\mathbb{1} \rangle \geq \langle K(0) \vee \cdots \vee K(j) \rangle$ as desired.

Each l_n^f is localization away from $LF(n+1)$ by Proposition 3.8, which is strongly dualizable by Theorem 2.3. The identity is localization away from zero, and the zero functor is localization away from LS^0; these are both strongly dualizable. Therefore the SDGSC holds. On the other hand, Corollary B.13 in [HS99] shows that there are no nonzero compact objects in $\mathcal{H}$, so the GSC fails. □

Question 4.5. Classify localizing subcategories of $\mathcal{H}$.

It seems likely that every localizing subcategory of $\mathcal{H}$ is a Bousfield class, and so these are in bijection with $2^{\mathbb{N}}$, but we have been unable to prove this.

5. The $E(n)$- and $K(n)$- local categories

5.1. **The $E(n)$-local category.** Recall that $E(n)$ has $\langle E(n)\rangle = \langle K(0) \vee K(1) \vee \cdots \vee K(n)\rangle$. In this section, fix $L = L_n = L_{E(n)} : \mathcal{S} \to \mathcal{S}$ and let $\mathcal{L}_n$ denote the local category. The functor L_n is smashing, and so by Theorem 2.1 each $LF(i)$ is compact in $\mathcal{L}_n$. Hovey and Strickland [HS99] have studied $\mathcal{L}_n$ in detail, and determine the localizing subcategories, smashing localizations, and Bousfield lattice of $\mathcal{L}_n$. We begin by recalling these results.

Lemma 5.1. *For $0 \le i \le n$, we have $LK(i) = K(i)$, and for $i > n$ we have $LK(i) = 0$.*

Proof. This follows from $\langle E(n)\rangle = \langle K(0) \vee K(1) \vee \cdots \vee K(n)\rangle$. □

Theorem 5.2. [HS99, Thm. 6.14] The lattice of localizing subcategories of $\mathcal{L}_n$, ordered by inclusion, is in bijection with the lattice of subsets of the set $\{0, 1, ..., n\}$, where a localizing subcategory S corresponds to

$$\{i \mid K(i) \in \mathsf{S}\}.$$

Corollary 5.3. Every localizing subcategory of $\mathcal{L}_n$ is a Bousfield class, in particular a localizing subcategory S is the Bousfield class

$$\bigvee \langle K(j) \mid K(j) \notin \mathsf{S}, 0 \le j \le n\rangle.$$

Corollary 5.4. For every $n \ge 0$, there is a lattice isomorphism

$$f_n : \mathsf{BL}(\mathcal{L}_n) \xrightarrow{\sim} 2^{\{0,1,\ldots,n\}}.$$

Proof. The isomorphism is given by

$$\langle X\rangle = \bigvee_{X \wedge K(i) \neq 0} \langle K(i)\rangle \mapsto \{i \mid X \wedge K(i) \neq 0\},$$

$$N \subseteq \{0, 1, ..., n\} \mapsto \bigvee_{i \in N} \langle K(i)\rangle.$$

□

Theorem 5.5. [HS99, Cor. 6.10] If $L' : \mathcal{L}_n \to \mathcal{L}_n$ is a smashing localization, then $L' = L_i = L_i^f$ for some $0 \le i \le n$ or $L' = 0$. Thus the GSC holds on $\mathcal{L}_n$.

Corollary 5.6. On $\mathcal{L}_n$, all of $\mathsf{LTC1_i}$, $\mathsf{LTC2_i}$, $\mathsf{LTC3_i}$ hold for all i, and GSC and SDGSC also hold.

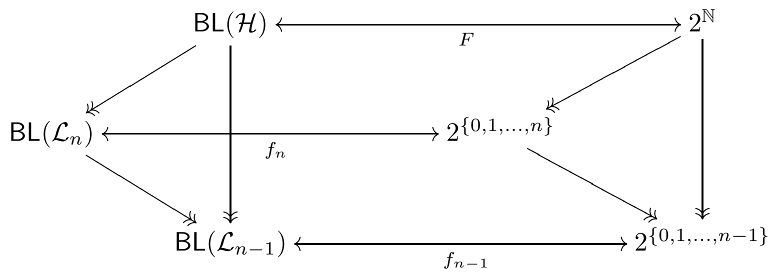
Proof. This follows from the previous Theorem, and Theorem 3.13. Note that for $i > n$, we have $LT(i) = 0 = LK(i)$ by Lemma 2.10, and so $l_i = l_n = l_n^f = l_i^f$. □

Recall that there is a natural map $L_n X \to L_{n-1} X$ for all X in $\mathcal{S}$ and n, and by Proposition 2.6 this induces a surjective lattice map $\mathsf{BL}(\mathcal{L}_n) \to \mathsf{BL}(\mathcal{L}_{n-1})$, and an inverse system of lattice maps.

$$\cdots \to \mathsf{BL}(\mathcal{L}_n) \to \mathsf{BL}(\mathcal{L}_{n-1}) \to \cdots \to \mathsf{BL}(\mathcal{L}_1) \to \mathsf{BL}(\mathcal{L}_0)$$

Proposition 5.7. The lattice isomorphisms F and f_n from Proposition 4.2 and Corollary 5.4 realize $\mathsf{BL}(\mathcal{H})$ as the inverse limit of the maps $\mathsf{BL}(\mathcal{L}_n) \to \mathsf{BL}(\mathcal{L}_{n-1})$.

Proof. From Lemma 2.8, and the facts that $L_Q K(i) = K(i)$ for all i, and $L_n K(i) = K(i)$ for $i \leq n$ and $L_n K(i) = 0$ for $i > n$, we get the following diagram for all n. The map $2^{\{0,1,\ldots,n\}} \to 2^{\{0,1,\ldots,n-1\}}$ is induced by sending $m \mapsto m$ for $m < n$ but $n \mapsto 0$, and the maps $2^{\mathbb{N}} \to 2^{\{0,1,\ldots,i\}}$ are defined similarly.



□

5.2. **The $K(n)$-local category.** Although an incredibly complicated category in its own right, the $K(n)$-local category is quite basic from the perspective of localizing subcategories, Bousfield lattices, and telescope conjectures. In this subsection, let $L = L_{K(n)} : \mathcal{S} \to \mathcal{S}$ be localization at $K(n)$, and let $\mathcal{K}_n$ denote the category of locals. Hovey and Strickland classify the localizing subcategories of $\mathcal{K}_n$, and there are not many of them.

Proposition 5.8. [HS99, Thm. 7.5] There are no nonzero proper localizing subcategories of $\mathcal{K}_n$.

This Proposition implies that the Bousfield lattice of $\mathcal{K}_n$ is the two-element lattice $\{\langle 0\rangle, \langle K(n)\rangle\}$. We will prove a slightly more general result, that will be used again in Subsection 6.1.

Proposition 5.9. Consider a category T as in Definition 2.1, an object Z in T, and localization $L_Z : \mathsf{T} \to \mathsf{T}$ with localized category $\mathcal{L}_Z$.

(1) If Z is a ring object, then $\langle L_Z Z\rangle = \langle Z\rangle$ is the maximum class in $\mathsf{BL}(\mathcal{L}_Z)$.
(2) If Z is a skew field object, then $\mathsf{BL}(\mathcal{L}_Z)$ is the two-element lattice $\{\langle 0\rangle, \langle Z\rangle\}$.

Proof. For (1), first note that Lemma 2.5 implies $L_Z Z = Z$. Consider $\langle Z\rangle$ in $\mathsf{BL}(\mathcal{L}_Z)$. By definition, this is the collection of all $W \in \mathcal{L}_Z$ with $L(Z \wedge_{\mathsf{T}} W) = 0$. But $Z \wedge_{\mathsf{T}} W$ is a Z-module object in T, hence is L_Z-local. The only object that is both local and acyclic with respect to any localization is zero, so $Z \wedge_{\mathsf{T}} W = 0$. But this says that W is L_Z-acyclic, hence zero in $\mathcal{L}_Z$. Therefore in $\mathsf{BL}(\mathcal{L}_Z)$ we have $\langle Z\rangle = \{0\}$.

Now suppose Z is a skew field object in T. In particular, it is a ring object, so $\langle Z\rangle$ is the maximum class in $\mathsf{BL}(\mathcal{L}_Z)$. Consider $\langle LX\rangle$ in $\mathsf{BL}(\mathcal{L}_Z)$, for arbitrary $X \in \mathsf{T}$. If $X \wedge_{\mathsf{T}} Z = 0$, then $LX = 0$. Otherwise, $X \wedge_{\mathsf{T}} Z$ is a wedge of suspensions of Z, so $\langle Z\rangle = \langle X \wedge_{\mathsf{T}} Z\rangle \leq \langle X\rangle$ in $\mathsf{BL}(\mathsf{T})$. Then $\langle Z\rangle = \langle L_Z Z\rangle \leq \langle L_Z X\rangle$ in $\mathsf{BL}(\mathcal{L}_Z)$, so $\langle L_Z X\rangle = \langle Z\rangle$. □

Corollary 5.10. The Bousfield lattice of $\mathcal{K}_n$ is $\{\langle 0\rangle, \langle K(n)\rangle\}$.

Theorem 5.11. In $\mathcal{K}_n$, all of $\mathsf{LTC1_i}$, $\mathsf{LTC2_i}$, $\mathsf{LTC3_i}$ hold for all i, and GSC and SDGSC also hold.

Proof. In light of Theorem 3.12, we will show that $\mathsf{LTC1_i}$ holds for all i. This follows from Lemma 2.10: for $i \neq n$ we have $LT(i) = 0 = LK(i)$, but $LT(n) \neq 0$ so by the last corollary $\langle LT(n)\rangle = \langle K(n)\rangle = \langle LK(n)\rangle$.

There are exactly two smashing localizations on $\mathcal{K}_n$. The identity functor is smashing, and is localization away from 0, which is compact and strongly dualizable. The zero functor is smashing, and is localization away from LS^0, which is strongly dualizable. It is not compact, but by Theorem 7.3 in [HS99], $LF(n)$ is compact in $\mathcal{K}_n$ and $\mathsf{loc}(LF(n)) = \mathsf{loc}(LS^0) = \mathcal{K}_n$. Therefore the zero functor is also generated by a compact object. This shows that both the GSC and SDGSC hold. □

6. Other localized categories

In this section we will consider several other localized categories. In each case, let $L_Z : \mathcal{S} \to \mathcal{S}$ denote the localization functor that annihilates $\langle Z\rangle$, and let $\mathcal{L}_Z$ denote the category of L_Z-locals.

6.1. **The $H\mathbb{F}_p$-local category.** The Eilenberg-MacLane spectrum $H\mathbb{F}_p$ is a skew field object in $\mathcal{S}$; in fact, every skew field object in $\mathcal{S}$ is isomorphic to either $H\mathbb{F}_p$ or a $K(n)$. Unlike the $\langle K(n)\rangle$, it is not complemented; for example, $\langle I\rangle \leq \langle H\mathbb{F}_p\rangle$ but $I \wedge H\mathbb{F}_p = 0$. So $\langle H\mathbb{F}_p\rangle \in \mathsf{DL}\backslash\mathsf{BA}$. Hovey and Palmieri [HP99] have conjectured several results about the collection of classes less than $\langle H\mathbb{F}_p\rangle$ in $\mathsf{BL}(\mathcal{S})$. The telescope conjectures and Bousfield lattice of $\mathcal{L}_{H\mathbb{F}_p}$ are quite simple.

Theorem 6.1. In $\mathcal{L}_{H\mathbb{F}_p}$, all of $\mathsf{LTC1_n}$, $\mathsf{LTC2_n}$, $\mathsf{LTC3_n}$ hold for all n.

Proof. For all n, $K(n) \wedge H\mathbb{F}_p = 0$ and $T(n) \wedge H\mathbb{F}_p = 0$, by [HP99, p. 16]. Therefore $LK(n) = 0 = LT(n)$ and $\mathsf{LTC1_n}$ holds for all n. Note that $l_n = l_n^f$ is the zero functor for all n. □

In order to discuss the GSC and SDGSC in this category, we must classify the smashing localizations. We will do this by using what we know about the Bousfield lattice.

Proposition 6.2. The Bousfield lattice of $\mathcal{L}_{H\mathbb{F}_p}$ is the two-element lattice $\{\langle 0\rangle, \langle H\mathbb{F}_p\rangle\}$.

Proof. This follows immediately from Proposition 5.9 because $H\mathbb{F}_p$ is a skew field object in $\mathcal{S}$. □

Recall that every smashing localization gives a pair of complemented classes in $\mathsf{BA} \subseteq \mathsf{BL}$. Thus in $\mathcal{L}_{H\mathbb{F}_p}$ there are exactly two smashing localizations, the trivial ones given by smashing with zero and with the unit.

Proposition 6.3. In $\mathcal{L}_{H\mathbb{F}_p}$, the GSC fails but the SDGSC holds.

Proof. The identity functor is smashing, and is localization away from 0, which is compact and strongly dualizable. By [HS99, Cor. B. 13], there are no nonzero compact objects in $\mathcal{L}_{H\mathbb{F}_p}$. So the zero functor, which is localization away from LS^0, is generated by a strongly dualizable object but not a compact one. □

One application of this Bousfield lattice calculation is to the question of classifying localizing subcategories. Every Bousfield class is a localizing subcategory.

In Conjecture 9.1 of [HP99], Hovey and Palmieri conjecture the converse holds, in the p-local stable homotopy category. The original conjecture is still open, but the question can be asked in any well-generated tensor triangulated category. For example, in a stratified category every localizing subcategory is a Bousfield class. The question is interesting, since in general localizing subcategories are hard to classify. In many cases, including $\mathcal{S}$, it is not even known if there is a set of localizing subcategories. Recently Stevenson [Ste12] found the first counterexample, in an algebraic setting: in the derived category of an absolutely flat ring that is not semi-artinian, there are localizing subcategories that are not Bousfield classes. Now we show that $\mathcal{L}_{H\mathbb{F}_p}$ provides another counterexample.

Proposition 6.4. In $\mathcal{L}_{H\mathbb{F}_p}$ there are localizing subcategories that are not Bousfield classes.

Proof. The following counterexample was suggested to us by Mark Hovey. The Bousfield lattice of $\mathcal{L}_{H\mathbb{F}_p}$ has only two elements: $\langle 0\rangle = \mathcal{L}_{H\mathbb{F}_p}$ and $\mathbb{1} = \{0\}$. It suffices to find a proper nonzero localizing subcategory in $\mathcal{L}_{H\mathbb{F}_p}$.

Consider the Moore spectrum $M(p)$, defined by the triangle $S^0 \xrightarrow{p} S^0 \to M(p)$; this spectrum is $H\mathbb{F}_p$-local. Consider the following full subcategory in $\mathcal{L}_{H\mathbb{F}_p}$.

$$\mathcal{A} = \{X \in \mathcal{L}_{H\mathbb{F}_p} \mid [X, M(p)]_n = 0 \text{ for all } n \in \mathbb{Z}\}.$$

This is a localizing subcategory, called the cohomological Bousfield class of $M(p)$ and denoted $\langle M(p)^*\rangle$ in [Hov95b]. The spectrum $H\mathbb{F}_p$ is a ring object, hence local with respect to itself. As mentioned in Section 4, it is known that $H\mathbb{F}_p$ is dissonant and $M(p)$ is harmonic, so $[H\mathbb{F}_p, M(p)]_n = 0$ for all n, and $H\mathbb{F}_p \in \mathcal{A}$. On the other hand, the identity on $M(p)$ is nonzero, so $M(p) \notin \mathcal{A}$. This shows that $\mathcal{A}$ is a localizing subcategory that is not a Bousfield class.

Another example comes from $Z = L_{H\mathbb{F}_p}(BP)$. Clearly $Z \notin \langle Z^*\rangle$. But BP is also harmonic, so $[H\mathbb{F}_p, BP]_n = 0$ and $[H\mathbb{F}_p, Z]_n = 0$ for all n, and $H\mathbb{F}_p \in \langle Z^*\rangle$. Since $Z \in \langle M(p)^*\rangle$, we know that $\langle M(p)^*\rangle \neq \langle Z^*\rangle$. □

Both these counterexamples are cohomological Bousfield classes. It would be interesting to find a localizing subcategory in $\mathcal{L}_{H\mathbb{F}_p}$ that is not a cohomological Bousfield class, or show there are none. Also, it is not clear what, if anything, the previous proposition might tell us about the original conjecture in $\mathcal{S}$. For example, as localizing subcategories in $\mathcal{S}$, we have [Hov95b, 3.3] that $\langle M(p)^*\rangle = \langle I\rangle$.

6.2. **The I-local category.** Recall by I we mean the Brown-Comenetz dual of the sphere spectrum. It is a rare example of a nonzero spectrum that squares to zero. Hovey and Palmieri [HP99, Lemma 7.8] conjecture that $\langle I\rangle$ is minimal in $\mathsf{BL}(\mathcal{S})$.

Theorem 6.5. On $\mathcal{L}_I$, for all n we have that $\mathsf{LTC1_n}$, $\mathsf{LTC2_n}$, and $\mathsf{LTC3_n}$ all hold.

Proof. By Lemma 7.1(c) of [HP99], $T(n)\wedge I = 0$ for all n, so $LT(n) = 0$. Since $K(i)$ is a BP-module, and $BP \wedge I = 0$ by [HS99, Cor. B.11], we also have $K(n) \wedge I = 0$ for all n. Therefore $\langle LT(n)\rangle = \langle 0\rangle = \langle LK(n)\rangle$ for all n, and the rest follows from Theorem 3.12. □

Proposition 6.6. The Bousfield lattice of $\mathcal{L}_I$ is the two-element lattice $\{\langle 0\rangle, \langle L_I S^0\rangle\}$.

Proof. By [HP99, 7.1(c)], $\langle I\rangle < \langle H\mathbb{F}_p\rangle$. Then Proposition 2.6 implies that there is a surjective lattice map from $\mathsf{BL}(\mathcal{L}_{H\mathbb{F}_p}) = \{\langle 0\rangle, \langle H\mathbb{F}_p\rangle\}$ onto $\mathsf{BL}(\mathcal{L}_I)$. Note that by Lemma 2.8, $\langle L_I H\mathbb{F}_p\rangle = \langle L_I L_{H\mathbb{F}_p} S^0\rangle = \langle L_I S^0\rangle$.

It remains to show that $\langle L_I S^0\rangle \neq \langle 0\rangle$. But any X in $\mathcal{S}$ with $X \wedge I \neq 0$ in $\mathcal{S}$ will have $L_I X \neq 0$ and $L_I X \notin \langle L_I S^0\rangle$ in $\mathsf{BL}(\mathcal{L}_I)$; this is because $L_I X \wedge_{\mathcal{L}_I} L_I S^0 = L_I(L_I X \wedge_{\mathcal{S}} L_I S^0) = L_I(X \wedge_{\mathcal{S}} S^0) = L_I(X)$. For example, $F(n) \wedge I \neq 0$ for all n [HP99, 7.1(e)]. □

Corollary 6.7. In $\mathcal{L}_I$, the GSC fails but the SDGSC holds.

Proof. Corollary B.13 of [HS99] also shows that $\mathcal{L}_I$ has no nonzero compacts, so the proof is the same as for $\mathcal{L}_{H\mathbb{F}_p}$. □

In [Hov95, Conj. 3.10], Hovey states the Dichotomy Conjecture: In $\mathcal{S}$ every spectrum has either a finite local or a finite acyclic. In [HP99] the authors discuss several equivalent formulations, and some implications. We briefly point out a relationship between this conjecture and Proposition 6.6.

Proposition 6.8. If the Dichotomy Conjecture holds, then the cardinality of $\mathsf{BL}(\mathcal{L}_I)$ is at most two.

Proof. Lemma 7.8 of [HP99] shows that if the Dichotomy Conjecture holds, then $\langle I\rangle$ is minimal among nonzero classes in $\mathsf{BL}(\mathcal{S})$. This is the case if and only if $a\langle I\rangle$ is maximal among non-top classes in $\mathsf{BL}(\mathcal{S})$, where $a(-)$ denotes the complementation operation first studied by Bousfield [Bou79b]. Let $a\langle I\rangle\!\uparrow$ denote the sublattice $\{\langle X\rangle \mid \langle X\rangle \geq a\langle I\rangle\} \subseteq \mathsf{BL}(\mathcal{S})$. In [Wol13, Prop. 3.2] we show that there is a surjective lattice map from $a\langle I\rangle\!\uparrow$ onto $\mathsf{BL}(\mathcal{L}_I)$. Thus, if the Dichotomy Conjecture holds, $a\langle I\rangle\!\uparrow$ has cardinality two and $\mathsf{BL}(\mathcal{L}_I)$ has cardinality at most two. □

As for classifying localizing subcategories of $\mathcal{L}_I$, or at least perhaps finding a proper nonzero localizing subcategory, we must get around the fact that so many spectra are I-acyclic. We know that $LF(n) \neq 0$ for all n, however $\mathsf{loc}(LF(n))$ is the acyclics of $l^f_{n-1} : \mathcal{L}_I \to \mathcal{L}_I$ and Theorem 6.5 shows that $l^f_n = 0$ for all n. Thus $\mathsf{loc}(LF(n)) = \mathsf{loc}(LS^0)$ in $\mathcal{L}_I$ for each n.

6.3. **The BP-local category.**

Theorem 6.9. On $\mathcal{L}_{BP}$, for all n we have that $\mathsf{LTC1_n}$, $\mathsf{LTC2_n}$, and $\mathsf{LTC3_n}$ all hold.

Proof. We will show that $\mathsf{LTC2_n}$ holds for all n, and the rest follows from Theorem 3.12. Since each $K(i)$ is a BP-module spectrum, $\langle K(i)\rangle \leq \langle BP\rangle$, and since $K(i)$ is local with respect to itself this implies that $K(i)$ is BP-local. Furthermore, this implies $\langle E(n)\rangle \leq \langle BP\rangle$, so from Lemma 2.8 $L_n = L_n L = L L_n$ as functors on $\mathcal{S}$.

We claim that $L_n : \mathcal{L}_{BP} \to \mathcal{L}_{BP}$, taking $LY \mapsto L_n LY = L_n Y$, is a smashing localization functor on $\mathcal{L}_{BP}$. We have $L_n(LY) = L(L_n Y) = L(L_n S^0 \wedge_{\mathcal{S}} Y) = L(LL_n S^0 \wedge_{\mathcal{S}} LY) = L(L_n LS^0 \wedge_{\mathcal{S}} LY) = (L_n LS^0) \wedge_{\mathcal{L}_{BP}} (LY)$. This shows that on $\mathcal{L}_{BP}$, the localization functor L_n is also given by smashing with the localization of the unit, $L_n LS^0$, and thus is smashing.

Since both L_n and l_n are localization functors on $\mathcal{L}_{BP}$ that annihilate $\langle K(0) \vee \cdots \vee K(n)\rangle = \langle LK(0) \vee \cdots \vee LK(n)\rangle$, they are isomorphic.

On $\mathcal{S}$, the natural map $L_n^f X \to L_n X$ is a BP-equivalence [Rav93, Thm. 2.7(iii)]. This means that $LL_n^f X = LL_n X$ for all objects X in $\mathcal{S}$, in particular for all BP-local objects. Therefore $LL_n^f = L_n = l_n$ is a smashing localization functor on $\mathcal{L}_{BP}$. The acyclics are $\langle LL_n^f(LS^0)\rangle = \langle LL_n^f S^0\rangle = \langle LT(0) \vee \cdots \vee LT(n)\rangle$. These are the same acyclics as for l_n^f, and so we conclude that l_n^f and l_n are isomorphic, and the natural map $l_n^f X \to l_n X$ is an isomorphism. □

Proposition 6.10. The GSC fails in $\mathcal{L}_{BP}$.

Proof. The proof of the last theorem showed that $L_n : \mathcal{L}_{BP} \to \mathcal{L}_{BP}$ is a (different) smashing localization for each n. However, by [HS99, Cor. B.13] the category $\mathcal{L}_{BP}$ has no nonzero compact objects. □

Note that the SDGSC could still hold, since all the smashing localizations we have identified on $\mathcal{L}_{BP}$ are of the form $L_n = l_n = l_n^f$, so are generated by strongly dualizable objects. The question of finding any other smashing localizations on $\mathcal{L}_{BP}$ is probably at least as hard as doing so on $\mathcal{S}$, in light of Proposition 3.16.

All of $\langle E(n)\rangle$, $\langle K(n)\rangle$, $\langle H\mathbb{F}_p\rangle$, and $\langle I\rangle$ are "small" in $\mathsf{BL}(\mathcal{S})$, so by Lemma 2.8 it is not surprising that the Bousfield lattices of their localized categories are not very large; this is not true of $\langle BP\rangle$ in $\mathsf{BL}(\mathcal{S})$. We have the following bounds on the Bousfield lattice of the local category.

Proposition 6.11. The Bousfield lattice of $\mathcal{L}_{BP}$ has $2^{\aleph_0} \leq |\mathsf{BL}(\mathcal{L}_{BP})| \leq 2^{2^{\aleph_0}}$.

Proof. The second inequality is Corollary 2.7. Since $\langle K(i)\rangle \leq \langle BP\rangle$ for all i, we have $\langle Q\rangle = \langle \bigvee_{i\geq 0} K(i)\rangle \leq \langle BP\rangle$, and so by Propositions 2.6 and 4.2 we have $|\mathsf{BL}(\mathcal{L}_{BP})| \geq |\mathsf{BL}(\mathcal{H})| = 2^{\aleph_0}$. □

6.4. **The $F(n)$-local category.** We conclude with a short discussion of the $F(n)$-local category.

Any smashing localization $L : \mathcal{S} \to \mathcal{S}$ gives a splitting of the Bousfield lattice

$$\mathsf{BL}(\mathcal{S}) \xrightarrow{\sim} \mathsf{BL}(\mathcal{L}_{LS^0}) \times \mathsf{BL}(\mathcal{L}_{CS^0}),$$

where $\langle X\rangle \mapsto (\langle X \wedge LS^0\rangle, \langle X \wedge CS^0\rangle)$. See [IK13, Prop. 6.12] or [Wol13, Thm. 5.14] for more details. If we take $L = L_n^f : \mathcal{S} \to \mathcal{S}$, then we have $\langle LS^0\rangle = \langle T(0) \vee \cdots \vee T(n)\rangle$ and $\langle CS^0\rangle = \langle F(n+1)\rangle$. Of course, the relationship between $\mathcal{L}_{T(0)\vee\cdots\vee T(n)}$ and $\mathcal{L}_{E(n)}$ of Subsection 5.1 is immediately related to the original $\mathsf{TC1_n}$ in $\mathcal{S}$. However, this suggests that $\mathcal{L}_{F(n)}$ is worth investigating further.

By Lemma 2.10, in $\mathsf{BL}(\mathcal{S})$ there is a chain

$$\langle S^0\rangle = \langle F(0)\rangle \geq \langle F(1)\rangle \geq \langle F(2)\rangle \geq \cdots,$$

and by Lemma 2.8 this gives a chain of lattice surjections

$$\mathsf{BL}(\mathcal{S}) = \mathsf{BL}(\mathcal{L}_{F(0)}) \twoheadrightarrow \mathsf{BL}(\mathcal{L}_{F(1)}) \twoheadrightarrow \mathsf{BL}(\mathcal{L}_{F(2)}) \twoheadrightarrow \cdots.$$

From the above observations, we expect $\mathsf{BL}(\mathcal{L}_{F(n)})$ to be about as complicated as $\mathsf{BL}(\mathcal{S})$. For example, $F(n) \wedge I \neq 0$ for all n, and so $L_{F(n)} I$ is a square-zero object in $\mathcal{L}_{F(n)}$. This means that, unlike in most of the localized categories discussed throughout this paper, we know that $\mathsf{BA}(\mathcal{L}_{F(n)}) \neq \mathsf{BL}(\mathcal{L}_{F(n)})$.

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