A Non-Noetherian Graded Mess: Some Bousfield Lattice Products and Quotients

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Bousfield Lattice

Let $\mathcal{T}$ be a well-generated tensor triangulated category. Denote the tensor product by $- \wedge -$ and the unit by 1.

Given objects X, Y, and W in $\mathcal{T}$, we say W is *X-acyclic* if $W \wedge X = 0$.

The collection of X-acyclics is denoted $\langle X \rangle$ and called the *Bousfield class* of X.

We say X and Y are *Bousfield equivalent* if $\langle X \rangle = \langle Y \rangle$.

The collection of all Bousfield classes is a complete lattice called the *Bousfield lattice*, and denoted $\mathrm{BL}_{\mathcal{T}}$.

The partial ordering is given by reverse inclusion:

$$\langle X \rangle \leq \langle Y \rangle \text{ if and only if } W \wedge Y = 0 \text{ implies } W \wedge X = 0.$$

Bousfield Lattice

The maximum class is $\langle 1\rangle$, the minimum is $\langle 0\rangle$, and the join is given by

$$\bigvee_{\alpha}\langle X_\alpha\rangle = \left\langle \coprod_{\alpha} X_\alpha \right\rangle .$$

Note that the tensor product gives another operation of Bousfield classes, by

$$\langle X\rangle \wedge \langle Y\rangle := \langle X \wedge Y\rangle.$$

This is a lower bound, but in general not the meet.

We say a class $\langle X\rangle$ is *complemented* if there exists a class $\langle X^c\rangle$ such that

$$\langle X\rangle \vee \langle X^c\rangle = \langle 1\rangle \text{ and } \langle X\rangle \wedge \langle X^c\rangle = \langle 0\rangle.$$

Bousfield lattice examples: $D(A)$, $\mathrm{StMod}(kG)$, the stable homotopy category.

Derived category of a graded ring

Let R be a graded commutative ring. Let $D(R)$ be the unbounded derived category of chain complexes of graded right R-modules. Objects in $D(R)$ are bigraded.

Let $f : R \to S$ be a homomorphism of graded rings. Then extension of scalars ($M \mapsto M \otimes_R S$) induces a functor on chain complexes, and a derived functor $f_\bullet : D(R) \to D(S)$. It has a right adjoint, $i_\bullet : D(S) \to D(R)$, the forgetful functor.

The functor $f_\bullet$ is exact, sends $R \mapsto S$ and compact objects to compact objects, and commutes with $- \wedge -$ and coproducts. The functor $i_\bullet$ is exact, and commutes with coproducts and products.

Operations on BL

Given a ring map $f : R \to S$, define $f_\bullet : \mathrm{BL}_R \to \mathrm{BL}_S$ and $i_\bullet : \mathrm{BL}_S \to \mathrm{BL}_R$ by

$$f_\bullet\langle X\rangle = \langle f_\bullet X\rangle \text{ and } i_\bullet\langle X\rangle = \langle i_\bullet X\rangle \text{ for all } X.$$

Lemma

These are well-defined, order-preserving operations on Bousfield lattices (i.e. are *conservative*), and preserve arbitrary joins.

The ring(s)

Fix a prime p, and integers $n_i > 1$. Fix the following graded rings and maps, with $\deg(x_i) = 2^i$.

$$\Gamma := \frac{\mathbb{Z}_{(p)}[x_1, x_2, x_3 \ldots]}{(x_1^{n_1}, x_2^{n_2}, x_3^{n_3} \ldots)}, \quad \Lambda := \frac{\mathbb{F}_p[x_1, x_2, x_3 \ldots]}{(x_1^{n_1}, x_2^{n_2}, x_3^{n_3} \ldots)}, \quad \Pi := \frac{\mathbb{Q}[x_1, x_2, x_3 \ldots]}{(x_1^{n_1}, x_2^{n_2}, x_3^{n_3} \ldots)}.$$

Let $f : \Gamma \to \Lambda$ be the obvious projection. Let $g : \Gamma \to \Pi$ be inclusion.

We will investigate

$$f_\bullet : D(\Gamma) \rightleftarrows D(\Lambda) : i_\bullet \text{ and } g_\bullet : D(\Gamma) \rightleftarrows D(\Pi) : i_\bullet.$$

A few more definitions

A triangulated subcategory is called *thick* if it is closed under retracts.

A triangulated subcategory is called *localizing* if it is closed under retracts and coproducts.

The smallest thick (respectively localizing) subcategory containing an object X is denoted $\mathrm{th}(X)$ (respectively $\mathrm{loc}(X)$).

Propositions

Although $D(\Gamma) \ncong \mathrm{loc}(i_\bullet\Lambda) \times \mathrm{loc}(i_\bullet\Pi)$ we have

Proposition A

$$\mathrm{BL}_{D(\Gamma)} \cong \mathrm{BL}_{\mathrm{loc}(i_\bullet\Lambda)} \times \mathrm{BL}_{\mathrm{loc}(i_\bullet\Pi)}.$$

Proposition B

$$\mathrm{BL}_{\mathrm{loc}(i_\bullet\Lambda)} \cong \mathrm{BL}_{D(\Lambda)}.$$

Question C

Is $\mathrm{BL}_{\mathrm{loc}(i_\bullet\Pi)} \cong \mathrm{BL}_{D(\Pi)}$?

The plan

1. Explain Proposition A.

2. Consider the case where $h : R \to S$ gives $h_\bullet i_\bullet \langle X \rangle = \langle X \rangle$ for all X.

3. Apply #2 to the above non-Noetherian rings, and Proposition B and Question C.

4. Apply #2 to a surjection onto a Noetherian ring.

Proposition A

Proposition A

$$\mathrm{BL}_{D(\Gamma)} \cong \mathrm{BL}_{\mathrm{loc}(i_\bullet \Lambda)} \times \mathrm{BL}_{\mathrm{loc}(i_\bullet \Pi)}.$$

In $D(\Gamma)$, $i_\bullet \Lambda$ can be represented by

$$\cdots \to 0 \to \Gamma \xrightarrow{p} \Gamma \to 0 \to \cdots,$$

and there is an exact triangle $\Gamma \xrightarrow{p} \Gamma \longrightarrow i_\bullet \Lambda$.

Lemma

The subcategory $\mathrm{th}(i_\bullet \Lambda)$ is a proper nontrivial thick subcategory of compact objects in $D(\Gamma)$.

Proposition A

Let $L : D(\Gamma) \to D(\Gamma)$ be finite localization at $\mathrm{th}(i_\bullet \Lambda)$. The L-acyclics are precisely $\mathrm{loc}(i_\bullet \Lambda)$.

Note that $i_\bullet \Pi$ is represented by

$$\mathrm{colim}(\Gamma \xrightarrow{p} \Gamma \xrightarrow{p} \Gamma \longrightarrow \cdots).$$

Lemma

The classes $\langle i_\bullet \Lambda \rangle$ and $\langle i_\bullet \Pi \rangle$ are a complemented pair.

Proposition A

Lemma

The functor L satisfies the following.

L-acyclics $= \text{loc}(i_\bullet \Lambda) = \langle i_\bullet \Pi \rangle = \langle L\Gamma \rangle$.

L-locals $= \text{loc}(i_\bullet \Pi) = \langle i_\bullet \Lambda \rangle$.

Given a smashing localization, like L, Iyengar and Krause (2011) showed there is always a splitting of the Bousfield lattice by acyclics and locals. Thus we have a splitting

$$\text{BL}_{D(\Gamma)} \cong \text{BL}_{\text{loc}(i_\bullet \Lambda)} \times \text{BL}_{\text{loc}(i_\bullet \Pi)}.$$

$\square$

A new setting

Let $h : R \to S$ be a map of rings. In this section, we assume

$$h_\bullet i_\bullet \langle X \rangle = \langle X \rangle \text{ for all } X.$$

Lemma

Let W, Y, and Z be objects in $D(S)$. The following are equivalent.

1. $h_\bullet i_\bullet \langle X \rangle = \langle X \rangle$ for all X,
2. $i_\bullet W \wedge i_\bullet Z = 0$ if and only if $i_\bullet (W \wedge Z) = 0$,
3. $i_\bullet \langle Y \wedge Z \rangle = \langle i_\bullet Y \rangle \wedge \langle i_\bullet Z \rangle$.

A quotient lattice

Define

$$J := \{\langle X\rangle \mid \langle h_\bullet X\rangle = \langle 0\rangle\} \text{ and } \langle M\rangle := \bigvee_{\langle Y\rangle \in J} \langle Y\rangle.$$

Then $J = \{\langle X\rangle \mid \langle X\rangle \leq \langle M\rangle\}$.

Proposition

J is a complete ideal in BL_R, and $h_\bullet$ induces a poset map

$$\mathrm{BL}_R/J \longrightarrow \mathrm{BL}_S.$$

If $h_\bullet i_\bullet\langle X\rangle = \langle X\rangle$ for all X, this is an isomorphism of posets.

Other nice things

In general, the functor $i_\bullet$ preserves joins on BL_S, so has a right adjoint $r : \mathrm{BL}_R \to \mathrm{BL}_S$.

Proposition

If $h_\bullet i_\bullet \langle X \rangle = \langle X \rangle$ for all X, then $h_\bullet = r$, so for all Y and Z

$$\langle i_\bullet Y \rangle \leq \langle Z \rangle \text{ if and only if } \langle Y \rangle \leq \langle h_\bullet Z \rangle.$$

Let BA denote the collection of complemented Bousfield classes.

Let DL denote the collection of $\langle X \rangle$ such that $\langle X \wedge X \rangle = \langle X \rangle$.

In general, we always have $\mathrm{BA} \subseteq \mathrm{DL} \subseteq \mathrm{BL}$.

Other nice things

BA = complemented. DL $= \{\langle X\rangle \mid \langle X \wedge X\rangle = \langle X\rangle\}$. BA $\subseteq$ DL $\subseteq$ BL.

Proposition

Suppose $h_\bullet i_\bullet \langle X\rangle = \langle X\rangle$ for all X. The following hold.

1. The map $h_\bullet$ sends DL_R onto DL_S, and the map $i_\bullet$ injects DL_S into DL_R.

2. The map $h_\bullet$ sends BA_R onto BA_S, and the map $i_\bullet$ injects BA_S into BA_R. If $\langle Y\rangle \in \mathrm{BA}_S$ has complement $\langle Y^c\rangle$, then $\langle i_\bullet Y\rangle$ has complement $\langle i_\bullet(Y^c)\rangle \vee \langle M\rangle$. In particular, $\langle i_\bullet S\rangle$ is complemented, with complement $\langle M\rangle$.

3. The map $h_\bullet$ induces an isomorphism between the image of DL_R in BL_R/J and DL_S, with inverse $i_\bullet$. Likewise with BA.

The plan

1. Explain Proposition A.

2. Consider the case where $h : R \to S$ gives $h_\bullet i_\bullet \langle X \rangle = \langle X \rangle$ for all X.

3. Apply #2 to the above non-Noetherian rings, and Proposition B and Question C.

4. Apply #2 to a surjection onto a Noetherian ring.

Recall

$$\Gamma := \frac{\mathbb{Z}_{(p)}[x_1, x_2, x_3 \dots]}{(x_1^{n_1}, x_2^{n_2}, x_3^{n_3} \dots)}, \quad \Lambda := \frac{\mathbb{F}_p[x_1, x_2, x_3 \dots]}{(x_1^{n_1}, x_2^{n_2}, x_3^{n_3} \dots)}, \quad \Pi := \frac{\mathbb{Q}[x_1, x_2, x_3 \dots]}{(x_1^{n_1}, x_2^{n_2}, x_3^{n_3} \dots)}.$$

Let $f : \Gamma \to \Lambda$ be the obvious projection. Let $g : \Gamma \to \Pi$ be inclusion.

Proposition A

$$\mathrm{BL}_{D(\Gamma)} \cong \mathrm{BL}_{\mathrm{loc}(i_\bullet \Lambda)} \times \mathrm{BL}_{\mathrm{loc}(i_\bullet \Pi)}.$$

Proposition B

$$\mathrm{BL}_{\mathrm{loc}(i_\bullet \Lambda)} \cong \mathrm{BL}_{D(\Lambda)}.$$

Question C

Is $\mathrm{BL}_{\mathrm{loc}(i_\bullet \Pi)} \cong \mathrm{BL}_{D(\Pi)}$?

Proposition B

Recall $f_\bullet : D(\Gamma) \to D(\Lambda)$.

Lemma

For all X, we have $f_\bullet i_\bullet X \cong X \oplus \Sigma X$, so $f_\bullet i_\bullet \langle X \rangle = \langle X \rangle$.

This implies, among other things, that $\mathrm{BL}_\Gamma / J \cong \mathrm{BL}_\Lambda$.

The complement of $\langle i_\bullet \Lambda \rangle$ in $D(\Gamma)$ is $\langle M \rangle$, but we know this is $\langle i_\bullet \Pi \rangle$. So we have

$$J = \{\langle X \rangle \mid \langle X \rangle \leq \langle i_\bullet \Pi \rangle\} = \{\langle X \rangle \mid X \in \mathrm{loc}(i_\bullet \Pi)\} \cong \mathrm{BL}_{\mathrm{loc}(i_\bullet \Pi)}.$$

Therefore $\mathrm{BL}_\Lambda \cong \mathrm{BL}_\Gamma / J \cong \mathrm{BL}_{\mathrm{loc}(i_\bullet \Lambda)}$. □

What about Question C?

A new example

Forget about Γ, Λ, and Π. From now on, suppose $h: R \to S$ is a surjection, and S is Noetherian.

The Bousfield lattice of $D(S)$ is wunderbar.

Given a prime ideal $\mathfrak{p} \in \operatorname{Spec} S$, let $S /\!\!/ \mathfrak{p}$ be the Koszul object, and $K(\mathfrak{p}) = S/\!\!/\mathfrak{p} \wedge S_{\mathfrak{p}}$. Then $\mathrm{BA}_S = \mathrm{DL}_S = \mathrm{BL}_S$ is the Boolean algebra on the classes $\langle K(\mathfrak{p}) \rangle$.
If $\mathfrak{p} = (z_1, ..., z_n)$, let $y_i = h^{-1}(z_i)$ be a choice of preimages, and define

$$R/\!\!/\tilde{\mathfrak{p}} = R/y_1 \wedge \cdots R/y_n \text{ and } K(\tilde{\mathfrak{p}}) = R/\!\!/\tilde{\mathfrak{p}} \wedge R_{h^{-1}(\mathfrak{p})}.$$

New objects

Lemma

For all $\mathfrak{p} \in \operatorname{Spec} S$ and all choices of $\tilde{\mathfrak{p}}$, we have $h_\bullet(R /\!\!/ \tilde{\mathfrak{p}}) = S /\!\!/ \mathfrak{p}$ and $h_\bullet K(\tilde{\mathfrak{p}}) = K(\mathfrak{p})$.

Recall $J = \{\langle X \rangle \mid \langle h_\bullet X \rangle = \langle 0 \rangle\}$. Consider the quotient and poset map $\mathrm{BL}_R / J \to \mathrm{BL}_S$.

Lemma

In BL_R / J, we have $\langle i_\bullet K(\mathfrak{p}) \rangle = \langle K(\tilde{\mathfrak{p}}) \rangle$ for every choice of $\tilde{\mathfrak{p}}$.

Proposition

For all X in $D(S)$, we have $h_\bullet i_\bullet \langle X \rangle = \langle X \rangle$.

Since $i_\bullet$ and $h_\bullet$ preserve joins, we can assume $\langle X \rangle = \langle K(\mathfrak{p}) \rangle$ for some $\mathfrak{p} \in \operatorname{Spec} S$. The lemmas imply $\langle h_\bullet i_\bullet K(\mathfrak{p}) \rangle = \langle h_\bullet K(\tilde{\mathfrak{p}}) \rangle = \langle K(\mathfrak{p}) \rangle$. $\square$

Crouching tiger, hidden dragon

This proposition allows us to apply the earlier results, to conclude that $h_\bullet$ induces an isomorphism $\mathrm{BL}_R/J \cong \mathrm{BL}_S$, with inverse $i_\bullet$. Recall that in BL_R/J we have $\langle i_\bullet K(\mathfrak{p})\rangle = \langle K(\tilde{\mathfrak{p}})\rangle$ for all $\mathfrak{p} \in \operatorname{Spec} S$.

Proposition

In BL_R/J the following hold.

1. If $\mathfrak{p} \neq \mathfrak{p}'$, then $\langle K(\tilde{\mathfrak{p}})\rangle \wedge \langle K(\tilde{\mathfrak{p}}')\rangle = \langle 0\rangle$.
2. $\langle K(\tilde{\mathfrak{p}})\rangle \wedge \langle K(\tilde{\mathfrak{p}})\rangle = \langle K(\tilde{\mathfrak{p}})\rangle$ for all $\tilde{\mathfrak{p}}$.
3. $\langle K(\tilde{\mathfrak{p}})\rangle$ is a minimal nonzero Bousfield class for all $\tilde{\mathfrak{p}}$.
4. $\langle R\rangle = \coprod_{\mathfrak{p}\in \operatorname{Spec} S}\langle K(\tilde{\mathfrak{p}})\rangle$.

Crouching tiger, hidden dragon

Most or all of these fail in BL_R, but we would like to understand the failure.

Note that if R is also Noetherian, $h_\bullet$ induces $\mathrm{BL}_R \cong \mathrm{BL}_S \times J$.

If, for an object X in $D(T)$, we define $\mathrm{supp}(X) = \{\mathfrak{p} \in \mathrm{Spec}\, T \mid X \wedge k_\mathfrak{p} \neq 0\}$, where $k_\mathfrak{p}$ is the residue field at $\mathfrak{p}$, then whether or not R is Noetherian we get the following.

Lemma

1. $\mathrm{supp}(i_\bullet X) = f^{-1}(\mathrm{supp}(X))$ for all X,
2. $\mathrm{supp}(i_\bullet(S/\!\!/\mathfrak{p})) = f^{-1}(V(\mathfrak{p})) = V(f^{-1}\mathfrak{p})$,
3. $f^{-1}(V(\mathfrak{p})) = V(\tilde{\mathfrak{p}}) \cap f^{-1}(\mathrm{Spec}\, S)$ for all choices of $\tilde{\mathfrak{p}}$,
4. $\mathrm{supp}(R/\!\!/\tilde{\mathfrak{p}}) = V(\tilde{\mathfrak{p}})$.

The end

Thank you.