Not every object in the derived category of a ring is Bousfield equivalent to a module

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Syrian landscape

Definitions and notation

Let k be a countable field, and define

$$\Lambda = \frac{k[x_1, x_2, \ldots]}{(x_1^2, x_2^2, \ldots)},$$

where $\deg(x_i) = 2^i$. Thus Λ is graded-connected and locally finite.

Let $D(\Lambda)$ be the unbounded derived category of chain complexes of graded Λ-modules (and graded maps). Use $X \wedge Y$ to denote $X \otimes^L_\Lambda Y$, and use $\pi_n(X)$ to denote $H_n(X)$.

For a Λ-module N, define $I(N) = \mathrm{Hom}^*_k(N, k)$ to be the graded vector space dual. This is again a graded Λ-module. We will consider modules as objects of $D(\Lambda)$, concentrated at chain degree zero.

Definitions and notation

Given an object X of $D(\Lambda)$, define the *X-acyclics* to be the collection $\{W \mid W \wedge X = 0\}$.

We say two objects X, Y are *Bousfield equivalent* if they have the same acyclics.

This gives an equivalence relation. The equivalence class of X is denoted $\langle X \rangle$, and called the *Bousfield class of X*.

Theorem statement

In this talk, we will construct an object Tel with $\pi_n(\mathrm{Tel}) \cong I(\Lambda)$ for all $n \in \mathbb{Z}$. We will use Tel to prove the following theorem.

Theorem 6.1.4

There are objects in $D(\Lambda)$ that are not Bousfield equivalent to a module. Specifically, there are $I(\Lambda)$-acyclics that are not Tel-acyclics, and any such object cannot be Bousfield equivalent to a module.

Noetherian case

On the other hand, if R is an ungraded Noetherian ring, and $D(R)$ is the derived category of ungraded modules, then every object *is* Bousfield equivalent to a module.

Specifically, for any object X of $D(R)$ we have

$$\langle X \rangle = \left\langle \bigoplus_{\mathfrak{p} \in \operatorname{supp}(X)} k_{\mathfrak{p}} \right\rangle,$$

where $\operatorname{supp}(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid k_{\mathfrak{p}} \wedge X \neq 0\}$ and $k_{\mathfrak{p}}$ is the residue field of $\mathfrak{p}$.

Bousfield background

We put a partial ordering on Bousfield classes, by saying $\langle X \rangle \leq \langle Y \rangle$ when

$$W \wedge Y = 0 \Longrightarrow W \wedge X = 0.$$

The Bousfield class $\langle \Lambda \rangle$ is a maximum, and $\langle 0 \rangle$ is a minimum. Furthermore, [DP08] showed that every nonzero Bousfield class $\langle X \rangle \neq \langle 0 \rangle$ has $\langle X \rangle \geq \langle I(\Lambda) \rangle$.

There is a set of Bousfield classes in $D(\Lambda)$. We have a join: $\langle X \rangle \vee \langle Y \rangle = \langle X \amalg Y \rangle$, a meet, and the operation $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. These make the set of Bousfield classes into a lattice, called the *Bousfield lattice*.

Definition of Tel

Let C in $D(\Lambda)$ be represented by the chain complex

$$0 \longrightarrow \underset{(0)}{\Lambda} \xrightarrow{x_1} \underset{(-1)}{\Lambda} \xrightarrow{x_1 x_2} \underset{(-2)}{\Lambda} \xrightarrow{x_2 x_3} \underset{(-3)}{\Lambda} \xrightarrow{x_3 x_4} \underset{(-4)}{\Lambda} \xrightarrow{x_4 x_5} \underset{(-5)}{\Lambda} \xrightarrow{x_5 x_6} \cdots$$

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Define a chain map $f : C \to \Sigma^2 C$ as follows.

$$\begin{array}{ccccccccccccccc}
 & & & & 0 & \longrightarrow & \Lambda & \xrightarrow{x_1} & \Lambda & \xrightarrow{x_1 x_2} & \Lambda & \xrightarrow{x_2 x_3} & \Lambda & \xrightarrow{x_3 x_4} & \cdots \\
 & & & & & & \downarrow{\scriptstyle x_3} & & \downarrow{\scriptstyle x_1 x_4} & & \downarrow{\scriptstyle x_2 x_5} & & \downarrow{\scriptstyle x_3 x_6} & & \\
0 & \longrightarrow & \Lambda & \xrightarrow[x_1]{} & \Lambda & \xrightarrow[x_1 x_2]{} & \Lambda & \xrightarrow[x_2 x_3]{} & \Lambda & \xrightarrow[x_3 x_4]{} & \Lambda & \xrightarrow[x_4 x_5]{} & \Lambda & \xrightarrow[x_5 x_6]{} & \cdots
\end{array}$$

Definition of Tel

Define Tel to be the sequential colimit

$$\mathrm{Tel} = \mathrm{colim}\left(C \xrightarrow{f} \Sigma^2 C \xrightarrow{\Sigma^2 f} \Sigma^4 C \longrightarrow \cdots\right).$$

This is a minimal weak colimit, so it satisfies

$$\begin{aligned}\pi_n(\mathrm{Tel}) &\cong \mathrm{colim}\left[\pi_n(C) \longrightarrow \pi_n(\Sigma^2 C) \longrightarrow \pi_n(\Sigma^4 C) \longrightarrow \cdots\right] \\ &\cong \mathrm{colim}\left[\pi_n(C) \longrightarrow \pi_{n-2}(C) \longrightarrow \pi_{n-4}(C) \longrightarrow \cdots\right].\end{aligned}$$

Proposition 6.1.1

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For all $n \in \mathbb{Z}$, the object Tel satisfies

$$\pi_n(\mathrm{Tel}) \cong I(\Lambda).$$

Proof. This proof will be deferred to the end of the talk. $\square$

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Proof. This proof will be deferred to the end of the talk. □

Given a Λ-module N, we defined the Λ-module $I(N) = \mathrm{Hom}_k^*(N, k)$ to be the graded dual. Now let $\mathbb{R}\mathrm{Hom}_k(-,-)$ denote $\mathbb{R}\mathrm{Hom}$ in the derived category of graded k-modules. For an arbitrary object X of $D(\Lambda)$, define $I(X) = \mathbb{R}\mathrm{Hom}_k(X, k)$, and think of this as an object in $D(\Lambda)$. Note that, when X is a module, this definition agrees with the previous one.

Lemma 6.1.2

Lemma 6.1.2

$$\langle \mathrm{Tel} \rangle \neq \langle I(\Lambda) \rangle.$$

Proof. First we show that $C \wedge I(\Lambda) \neq 0$. From tensor-hom adjointness at the module level, we get for all X and Y in $D(\Lambda)$:

$$\mathbb{R}\mathrm{Hom}_k(X \wedge Y, k) \cong \mathbb{R}\mathrm{Hom}_\Lambda(X, \mathbb{R}\mathrm{Hom}_k(Y, k)).$$

In particular, setting $Y = \Lambda$ gives $I(X) = \mathbb{R}\mathrm{Hom}_k(X, k) \cong \mathbb{R}\mathrm{Hom}_\Lambda(X, I(\Lambda))$ for all X in $D(\Lambda)$. Using this, we compute

$$\begin{aligned} I(C \wedge I(\Lambda)) &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C \wedge I(\Lambda), I(\Lambda)) \\ &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C, \mathbb{R}\mathrm{Hom}_\Lambda(I(\Lambda), I(\Lambda))) \\ &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C, I(I(\Lambda))) \\ &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C, \Lambda). \end{aligned}$$

Lemma 6.1.2

We have

$$\pi_0(\mathbb{R}\mathrm{Hom}_\Lambda(C, \Lambda)) \cong [C, \Lambda]_0.$$

The module Λ is self-injective, because Λ is a P-algebra, so $[C, \Lambda]_0$ is homotopy classes of degree zero chain maps from C to Λ.

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \Lambda & \xrightarrow[x_1]{} & \Lambda & \xrightarrow[x_1 x_2]{} & \Lambda & \xrightarrow[x_2 x_3]{} & \Lambda & \xrightarrow[x_3 x_4]{} & \Lambda & \xrightarrow[x_4 x_5]{} & \Lambda & \xrightarrow[x_5 x_6]{} & \cdots . \\
 & & \downarrow & & & & & & & & & & & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & 0 & & & & & & & & & &
\end{array}$$

There are nontrivial such maps, so $I(C \wedge I(\Lambda)) \neq 0$. Thus $C \wedge I(\Lambda) \neq 0$.

Lemma 6.1.2

Let K be the cofiber of $f : C \to \Sigma^2 C$. Then we have

$$\langle C \rangle = \langle K \rangle \vee \langle \mathrm{Tel} \rangle \text{ and } \langle 0 \rangle = \langle K \rangle \wedge \langle \mathrm{Tel} \rangle.$$

Now suppose that $\langle \mathrm{Tel} \rangle = \langle I(\Lambda) \rangle$. Then $\langle \mathrm{Tel} \rangle \leq \langle K \rangle$, so

$$\langle C \rangle = \langle K \rangle \vee \langle \mathrm{Tel} \rangle = \langle K \rangle.$$

This implies $\langle 0 \rangle = \langle C \rangle \wedge \langle \mathrm{Tel} \rangle = \langle C \wedge \mathrm{Tel} \rangle$, so $C \wedge \mathrm{Tel} = 0$.

This would force $C \wedge I(\Lambda) = 0$, which is a contradiction. $\square$

Lemma 6.1.3

Let $\mathcal{M}$ denote the full subcategory of $D(\Lambda)$ of all modules.

Lemma 6.1.3

$$\mathcal{M} \cap \{\text{Tel-acyclics}\} = \mathcal{M} \cap \{I(\Lambda)\text{-acyclics}\}.$$

Proof. Since $\langle I(\Lambda) \rangle$ is minimum among nonzero Bousfield classes, we know $\langle I(\Lambda) \rangle \leq \langle \text{Tel} \rangle$, so we already have the $\subseteq$ direction.

We will show that if M is a module in $D(\Lambda)$ and $M \wedge I(\Lambda) = 0$, then $M \wedge \text{Tel} = 0$.

Lemma 6.1.3

Kriz and May [KM95, III.4.7] construct a strongly convergent “Eilenberg-Moore" spectral sequence

$$E^2_{p,q} = \bigoplus_{m+n=q} \mathrm{Tor}^{\Lambda}_p(\pi_m(X), \pi_n(Y)) \Longrightarrow \pi_{p+q}(X \wedge Y).$$

Let $X = M$ be a Λ-module, and $Y = \mathrm{Tel}$, and use Proposition 6.1.1 to get

$$E^2_{p,q} = \mathrm{Tor}^{\Lambda}_p(M, I(\Lambda)) = \pi_p(M \wedge I(\Lambda)) \Longrightarrow \pi_{p+q}(M \wedge \mathrm{Tel}).$$

If $M \wedge I(\Lambda) = 0$ then the E^2 page collapses to zero, and $\pi_*(M \wedge \mathrm{Tel}) = 0$. □

Theorem 6.1.4

Now we can use these lemmas to prove the theorem.

Theorem 6.1.4

There are objects in $D(\Lambda)$ that are not Bousfield equivalent to a module. Specifically, there are $I(\Lambda)$-acyclics that are not Tel-acyclics, and any such object cannot be Bousfield equivalent to a module.

Proof. Suppose, towards a contradiction, that every object Y in $D(\Lambda)$ is Bousfield equivalent to some module, M_Y. Take X with $X \wedge I(\Lambda) = 0$. Then $M_X \wedge I(\Lambda) = 0$. Using Lemma 6.1.3, this says that

$$M_X \in \mathcal{M} \cap \{I(\Lambda)\text{-acyclics}\} = \mathcal{M} \cap \{\text{Tel-acyclics}\}.$$

Thus $M_X \wedge \text{Tel} = 0$, so $X \wedge \text{Tel} = 0$.
This implies that $\langle I(\Lambda)\rangle \geq \langle \text{Tel}\rangle$. Since we already have $\langle I(\Lambda)\rangle \leq \langle \text{Tel}\rangle$, we conclude that $\langle I(\Lambda)\rangle = \langle \text{Tel}\rangle$. But this contradicts Lemma 6.1.2. □

Questions

Question 6.1.5

We have shown that $\langle I(\Lambda)\rangle < \langle \mathrm{Tel}\rangle$, so that there are $I(\Lambda)$-acyclics that are not Tel-acyclic. Can we construct such an object more explicitly?

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Question 6.1.6

To what extent can this argument and result be applied to other derived categories, or to the stable homotopy category $\mathcal{S}$ of p-local spectra?

Consequences

- answers two questions posed in [DP08].
- shows that several approaches to classifying subcategories (Neeman's [Nee92] and [Nee11], stratification [BIK11]) will not work in the derived category of a non-Noetherian ring.
- another difference between the Noetherian and non-Noetherian contexts ([Nee00], [DP08]).
- matters because non-Noetherian rings show up a lot in algebraic topology [Pat11].

Proposition 6.1.1

Recall we defined the chain map $f : C \to \Sigma^2 C$ as follows.

$$\begin{array}{ccccccccccccccc}
 & & & & 0 & \longrightarrow & \Lambda & \xrightarrow{x_1} & \Lambda & \xrightarrow{x_1 x_2} & \Lambda & \xrightarrow{x_2 x_3} & \Lambda & \xrightarrow{x_3 x_4} & \cdots \\
 & & & & & & \downarrow{\scriptstyle x_3} & & \downarrow{\scriptstyle x_1 x_4} & & \downarrow{\scriptstyle x_2 x_5} & & \downarrow{\scriptstyle x_3 x_6} & & \\
0 & \longrightarrow & \Lambda & \xrightarrow[x_1]{} & \Lambda & \xrightarrow[x_1 x_2]{} & \Lambda & \xrightarrow[x_2 x_3]{} & \Lambda & \xrightarrow[x_3 x_4]{} & \Lambda & \xrightarrow[x_4 x_5]{} & \Lambda & \xrightarrow[x_5 x_6]{} & \cdots
\end{array}$$

And then we defined

$$\mathrm{Tel} = \mathrm{colim}\left(C \xrightarrow{f} \Sigma^2 C \xrightarrow{\Sigma^2 f} \Sigma^4 C \longrightarrow \cdots \right).$$

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For concreteness, we illustrate for $\pi_{-2}(\mathrm{Tel})$, and then discuss the general case. We will split the computation into several lemmas.

Recall that

$$\pi_n(\mathrm{Tel}) \cong \operatorname{colim}\left[\pi_n(C) \longrightarrow \pi_{n-2}(C) \longrightarrow \pi_{n-4}(C) \longrightarrow \cdots\right].$$

Proposition 6.1.1

We have

$$\pi_{-2}(C) = \frac{\ker(x_2x_3)}{\operatorname{im}(x_1x_2)} \cong \frac{(x_2, x_3)}{(x_1x_2)}, \text{ and generally } \pi_{-n}(C) \cong \frac{(x_n, x_{n+1})}{(x_{n-1}x_n)}, \text{ for } n \geq 2.$$

Define

$$M_{-2} = \frac{(x_3)}{(x_2, x_4, x_5, x_6, \dots)}, \text{ and in general } M_{-n} = \frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, x_{n+4}, \dots)},$$

and consider the collection of maps

$$\frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, \dots)} = M_{-n} \xrightarrow{x_nx_{n+3}} M_{-n-2} = \frac{(x_{n+3})}{(x_{n+2}, x_{n+4}, x_{n+5}, \dots)}.$$

Proposition 6.1.1

Lemma 6.1.7

$$\pi_{-2}(\mathrm{Tel}) \cong \operatorname{colim}\left[\frac{(x_2, x_3)}{(x_1 x_2)} \xrightarrow{x_2 x_5} \frac{(x_4, x_5)}{(x_3 x_4)} \xrightarrow{x_4 x_7} \frac{(x_6, x_7)}{(x_5 x_6)} \xrightarrow{x_6 x_9} \cdots\right]$$
$$\cong \operatorname{colim}\left[M_{-2} \xrightarrow{x_2 x_5} M_{-4} \xrightarrow{x_4 x_7} M_{-6} \xrightarrow{x_6 x_9} \cdots\right].$$

Proposition 6.1.1

Lemma 6.1.7

$$\pi_{-2}(\mathrm{Tel}) \cong \operatorname{colim}\left[\frac{(x_2,x_3)}{(x_1x_2)} \xrightarrow{x_2x_5} \frac{(x_4,x_5)}{(x_3x_4)} \xrightarrow{x_4x_7} \frac{(x_6,x_7)}{(x_5x_6)} \xrightarrow{x_6x_9} \cdots\right]$$
$$\cong \operatorname{colim}\left[M_{-2} \xrightarrow{x_2x_5} M_{-4} \xrightarrow{x_4x_7} M_{-6} \xrightarrow{x_6x_9} \cdots\right].$$

Lemma 6.1.8

$$\operatorname{colim}\left[M_{-2} \xrightarrow{x_2x_5} M_{-4} \xrightarrow{x_4x_7} M_{-6} \xrightarrow{x_6x_9} \cdots\right]$$
$$\cong \operatorname{colim}\left[I\left(\frac{k[x_1]}{(x_1^2)}\right) \hookrightarrow I\left(\frac{k[x_1,x_2,x_3]}{(x_1^2,x_2^2,x_3^2)}\right) \hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_1^2,x_2^2,x_3^2,x_4^2,x_5^2)}\right) \hookrightarrow \cdots\right].$$

Proposition 6.1.1

Lemma 6.1.9

$$\operatorname{colim}\left[I\left(\frac{k[x_1]}{(x_1^2)}\right) \hookrightarrow I\left(\frac{k[x_1,x_2,x_3]}{(x_1^2,x_2^2,x_3^2)}\right) \hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_1^2,x_2^2,x_3^2,x_4^2,x_5^2)}\right) \hookrightarrow \cdots\right]$$

$$\cong I\left(\lim\left[\cdots \to \frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_1^2,x_2^2,x_3^2,x_4^2,x_5^2)} \xrightarrow{proj} \frac{k[x_1,x_2,x_3]}{(x_1^2,x_2^2,x_3^2)} \xrightarrow{proj} \frac{k[x_1]}{(x_1^2)}\right]\right) \cong I(\Lambda).$$

These three lemmas together show that $\pi_{-2}(\mathrm{Tel}) \cong I(\Lambda)$, and in fact show $\pi_n(\mathrm{Tel}) \cong I(\Lambda)$ for all even n. By shifting indices, we get $\pi_n(\mathrm{Tel}) \cong I(\Lambda)$ for all odd n as well.

Lemma 6.1.7

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$$\pi_{-2}(\mathrm{Tel}) \cong \operatorname{colim}\left[\frac{(x_2,x_3)}{(x_1x_2)} \xrightarrow{x_2x_5} \frac{(x_4,x_5)}{(x_3x_4)} \xrightarrow{x_4x_7} \frac{(x_6,x_7)}{(x_5x_6)} \xrightarrow{x_6x_9} \cdots\right]$$
$$\cong \operatorname{colim}\left[M_{-2} \xrightarrow{x_2x_5} M_{-4} \xrightarrow{x_4x_7} M_{-6} \xrightarrow{x_6x_9} \cdots\right].$$

$$\begin{array}{ccc}
\pi_{-r}(C) \cong \dfrac{(x_r,x_{r+1})}{(x_{r-1}x_r)} & \xrightarrow{\quad x_rx_{r+3}\quad} & \dfrac{(x_{r+2},x_{r+3})}{(x_{r+1}x_{r+2})} \cong \pi_{-r-2}(C) \\
\downarrow \text{proj} & & \downarrow \text{proj} \\
M_{-r} = \dfrac{(x_{r+1})}{(x_r,x_{r+2},x_{r+3},\dots)} & \xrightarrow{\quad x_rx_{r+3}\quad} & \dfrac{(x_{r+3})}{(x_{r+2},x_{r+4},x_{r+5},\dots)} = M_{-r-2}
\end{array}$$

Lemma 6.1.8

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$$\operatorname{colim}\left[M_{-2} \xrightarrow{x_2 x_5} M_{-4} \xrightarrow{x_4 x_7} M_{-6} \xrightarrow{x_6 x_9} \cdots\right]$$

$$\cong \operatorname{colim}\left[I\left(\frac{k[x_1]}{(x_1^2)}\right) \hookrightarrow I\left(\frac{k[x_1, x_2, x_3]}{(x_1^2, x_2^2, x_3^2)}\right) \hookrightarrow I\left(\frac{k[x_1, x_2, x_3, x_4, x_5]}{(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2)}\right) \hookrightarrow \cdots\right].$$

$$\begin{array}{ccc}
M_{-r} = \dfrac{(x_{r+1})}{(x_r, x_{r+2}, x_{r+3}, \ldots)} & \xrightarrow{\quad x_r x_{r+3} \quad} & \dfrac{(x_{r+3})}{(x_{r+2}, x_{r+4}, x_{r+5}, \ldots)} = M_{-r-2} \\
\downarrow \cong & & \downarrow \cong \\
I\left(\dfrac{k[x_1, x_2, \ldots, x_{r-1}]}{(x_1^2, x_2^2, \ldots, x_{r-1}^2)}\right) & \xrightarrow{\quad \text{inclusion} \quad} & I\left(\dfrac{k[x_1, x_2, \ldots, x_{r+1}]}{(x_1^2, x_2^2, \ldots, x_{r+1}^2)}\right)
\end{array}$$

Dedicated to Eina Ooka

"Replace the rent with the stars above.
Replace the need with the love.
Replace the anger with the tide.
Replace the ones that you love.

Are you on fire, from the years?
What would you give for your kid fears?"

- "Kid Fears" by the Indigo Girls

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