Ring Maps, Derived Categories, and the Bousfield Lattice

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Introduction

Let R be a commutative ring.

Let $D(R)$ be the unbounded derived category.

Objects are chain complexes of (right) R-modules.

Morphisms are equivalence classes of chain maps, where quasi-isomorphisms are inverted.

D(R) is a monogenic stable homotopy category

$D(R)$ is nice:

- triangulated
- symmetric monoidal: $-\otimes^L_R -$ is tensor product, $\mathbb{R}Hom(-,-)$ gives function objects
- arbitrary coproducts exist
- unit of tensor product (R concentrated in degree zero) is a small, weak generator
- Brown Representability holds.

$D(R)$ is a monogenic stable homotopy category, like *Spectra* from topology, and $\mathcal{C}((kG)^*)$ from rep'n theory.

Interested in subcategories

- A triangulated subcategory is thick if it is closed under retracts.
- A triangulated subcategory is localizing if it is closed under retracts and arbitrary coproducts.
- An object is finite (or small, or compact) if it is in the thick subcategory generated by the tensor unit.

We're interested in localizing subcategories, and thick subcategories of finite objects.

Interested in the Bousfield lattice

- The Bousfield class of an object X is $\langle X \rangle = \{W \mid X \otimes W = 0\}$. This is an equivalence relation.
- The Bousfield lattice $\mathrm{BL}_{D(R)}$ is the collection of Bousfield classes.

Every Bousfield class is a localizing subcategory. So we're interested in the Bousfield lattice of $D(R)$.

When R is Noetherian we know a lot [Nee92, HPS97]

- Localizing subcategories of $D(R)$ are in bijection with arbitrary subsets of Spec R.
- Thick subcategories of finite objects of $D(R)$ are in bijection with specialization-closed subsets of Spec R.

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- Localizing subcategories of $D(R)$ are in bijection with arbitrary subsets of Spec R.
- Thick subcategories of finite objects of $D(R)$ are in bijection with specialization-closed subsets of Spec R.
- Every localizing subcategory is a Bousfield class.

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- Thick subcategories of finite objects of $D(R)$ are in bijection with Thomason-closed subsets of Spec R [Tho97].

When R is not necessarily Noetherian, less is known

- Thick subcategories of finite objects of $D(R)$ are in bijection with Thomason-closed subsets of $\operatorname{Spec} R$ [Tho97].
- In [DP08], authors investigate a specific non-Noetherian ring Λ. They show the Bousfield lattice of $D(\Lambda)$ has cardinality $2^{2^{\aleph_0}}$, although $|\operatorname{Spec} \Lambda| = 1$.

Our setup

Given a map $f : R \to S$ of commutative rings, extension by scalars

$$f_* : \text{Mod-}R \to \text{Mod-}S$$
$$M \mapsto M \otimes_R S$$

induces $f_\bullet : D(R) \to D(S)$,

Our setup

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and the forgetful functor $i_* : \text{Mod-}S \to \text{Mod-}R$ induces $i_\bullet : D(S) \to D(R)$.

The functors $f_\bullet$ and $i_\bullet$ are adjoints.

Research questions:

- How do $f_\bullet$ and $i_\bullet$ relate the subcategories and Bousfield lattices of $D(R)$ and $D(S)$?

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- How do $f_\bullet$ and $i_\bullet$ relate the subcategories and Bousfield lattices of $D(R)$ and $D(S)$?
- Can this be used to get information about $D(R)$ for an arbitrary commutative R?

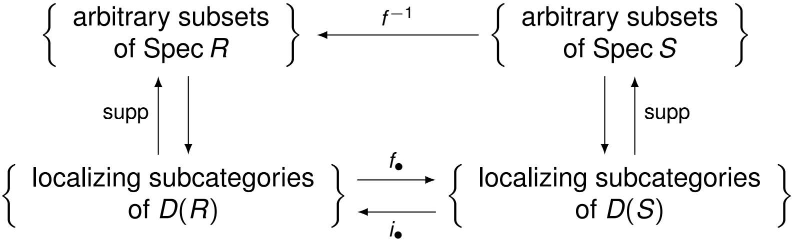
Surjection between Noetherian rings - localizing subcategories

Suppose R and S are Noetherian, and $f : R \twoheadrightarrow S$ is a surjection.

Let $\mathcal{A}$ be a localizing subcategory of $D(R)$.

Define $f_{\bullet}\mathcal{A}$ to be the smallest localizing subcategory of $D(S)$ containing $f_{\bullet}X$ for all X in $\mathcal{A}$.

Surjection between Noetherian rings - localizing subcategories



Surjection between Noetherian rings - localizing subcategories

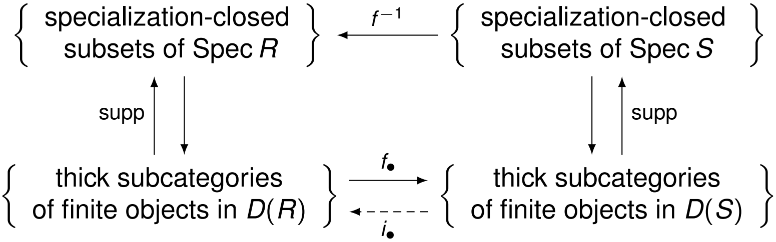
Propositions 6.1 and 6.5

Let $f : R \twoheadrightarrow S$ be a surjection between Noetherian rings. Let $\mathcal{A}$ and $\mathcal{B}$ be localizing subcategories of $D(R)$ and $D(S)$, respectively. Then

$$\operatorname{supp}(f_\bullet \mathcal{A}) = (f^{-1})^{-1}(\operatorname{supp} \mathcal{A})$$

$$\operatorname{supp}(i_\bullet \mathcal{B}) = f^{-1}(\operatorname{supp} \mathcal{B}).$$

Surjection between Noetherian rings - thick subcategories



Surjection between Noetherian rings - thick subcategories

Propositions 5.22 and 6.9

Let $f : R \twoheadrightarrow S$ be a surjection between Noetherian rings. Let $\mathcal{A}$ and $\mathcal{B}$ be thick subcategories of finite objects in $D(R)$ and $D(S)$, respectively. Then

$$\operatorname{supp}(f_\bullet \mathcal{A}) = (f^{-1})^{-1}(\operatorname{supp} \mathcal{A}).$$

In general, $i_\bullet$ doesn't send finite objects to finite objects, but when it does, we also have

$$\operatorname{supp}(i_\bullet \mathcal{B}) = f^{-1}(\operatorname{supp} \mathcal{B}).$$

Surjection between Noetherian rings - Bousfield lattice

Take X in $D(R)$ and Y in $D(S)$. Define operations

$$f_{\bullet}\langle X\rangle = \langle f_{\bullet}X\rangle,$$

$$i_{\bullet}\langle Y\rangle = \langle i_{\bullet}Y\rangle.$$

These give well-defined, order-preserving operations between $\mathrm{BL}_{D(R)}$ and $\mathrm{BL}_{D(S)}$.

Surjection between Noetherian rings - Bousfield lattice

Define $J = \{\langle X \rangle \text{ such that } f_\bullet \langle X \rangle = \langle 0 \rangle\}$.

Proposition

Suppose $f : R \twoheadrightarrow S$ is a surjection between Noetherian rings. Then $f_\bullet$ induces an isomorphism of lattices

$$\mathrm{BL}_{D(R)} \cong \mathrm{BL}_{D(S)} \times J.$$

More general surjection - Bousfield lattice

Proposition 4.2

Suppose $f : R \twoheadrightarrow S$ is a surjection, with R arbitrary and S Noetherian. Then $f_\bullet$ induces an isomorphism

$$\begin{array}{ccc} \mathrm{BL}_{D(R)} & \xrightarrow{f_\bullet} & \mathrm{BL}_{D(S)} \\ \downarrow & \nearrow_{\cong} & \\ \mathrm{BL}_{D(R)}/J & & \end{array}$$

.

More general surjection - other results

- No localizing subcategory results.
- Weaker version of thick subcategory correspondence.
- There are specific elements (Koszul objects) in $D(S)$ that play an important role, and these can be pulled back to $D(R)$.

More general surjection - Bousfield lattice 2

- The distributive lattice DL in BL is $\{\langle X\rangle$ such that $\langle X \otimes X\rangle = \langle X\rangle\}$.
- A Bousfield class $\langle X\rangle$ is complemented if there is a class $\langle X^c\rangle$ such that $\langle X \vee X^c\rangle = \langle R\rangle$ and $\langle X \wedge X^c\rangle = \langle 0\rangle$.
- The collection of complemented Bousfield classes is a Boolean algebra, denoted BA.
- In general, $BA \subseteq DL \subseteq BL$.

More general surjection - Bousfield lattice 3

Proposition 4.6

Suppose $f : R \twoheadrightarrow S$ is a surjection, with R arbitrary and S Noetherian. Then

- $f_\bullet$ sends $\mathrm{DL}_{D(R)}$ and $\mathrm{BA}_{D(S)}$ onto $\mathrm{DL}_{D(S)}$ and $\mathrm{BA}_{D(S)}$, respectively.
- $i_\bullet$ injects $\mathrm{DL}_{D(S)}$ and $\mathrm{BA}_{D(S)}$ into $\mathrm{DL}_{D(R)}$ and $\mathrm{BA}_{D(S)}$, respectively.

Example with non-Noetherian rings

Fix integers $n_i > 1$. Let p be a prime, k be the finite field of order p, and $\mathbb{Z}_{(p)}$ be the p-local integers. Define

$$\Lambda = \frac{k[x_1, x_2, x_3, \ldots]}{(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, \ldots)}, \text{ and } \Gamma = \frac{\mathbb{Z}_{(p)}[x_1, x_2, x_3, \ldots]}{(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, \ldots)}.$$

The category $D(\Lambda)$ is studied extensively in [DP08].
Let $f : \Gamma \twoheadrightarrow \Lambda = \Gamma/p\Gamma$ be the natural projection.

- We have used $f_\bullet$ and $i_\bullet$ to extend results from [DP08].
- For example, we show

$$\mathrm{BL}_{D(\Gamma)} \cong \mathrm{BL}_{D(\Lambda)} \times J,$$

and have a good description of J.

References

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HPS97 M.Hovey, J.H.Palmieri, and N.P.Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math.Soc. **128** (1997), no. 610, x+114. MR 98a:55017

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Tho97 R. W. Thomason, *The classification of triangulated subcategories*, Compositio Math., 105(1):1-27, 1997.