NOT EVERY OBJECT IN THE DERIVED CATEGORY OF A RING IS BOUSFIELD EQUIVALENT TO A MODULE

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Abstract. We consider the unbounded derived category of a specific non-Noetherian ring Λ, and show that there are objects in $D(\Lambda)$ that are not Bousfield equivalent to any module. This answers a question posed by Dwyer and Palmieri.

1. Introduction

Let k be a field, and consider the graded ring

$$\Lambda := \frac{k[x_1, x_2, x_3 ...]}{(x_1^2, x_2^2, x_3^2 ...)}, \text{ where } \deg(x_i) = 2^i.$$

Let mod-Λ denote the category of $\mathbb{Z}$-graded Λ-modules. We consider the unbounded derived category $D(\Lambda)$. An object in $D(\Lambda)$ has two gradings, one homological and one induced by the grading on modules.

The category $D(\Lambda)$ is a compactly generated tensor triangulated category; in fact it is a monogenic stable homotopy category, in the sense of [HPS97]. It has arbitrary coproducts and products, and has a symmetric monoidal product $-\otimes^L_\Lambda -$, which we will denote $- \wedge -$.

Given an object X of $D(\Lambda)$, define the *X-acyclics* to be the collection of all objects W with $W \wedge X = 0$. We say two objects X and Y are *Bousfield equivalent* if they have the same acyclics. This gives an equivalence relation on $D(\Lambda)$. The equivalence class of X is denoted $\langle X \rangle$, and called the *Bousfield class* of X. The collection of Bousfield classes forms a complete lattice, called the *Bousfield lattice*. (In Section 2 we give a more thorough background.)

The Bousfield lattice BL_Λ of $D(\Lambda)$ has been shown to exhibit interesting behavior, particularly in contrast with the Bousfield lattice of the derived category of a commutative Noetherian ring. For example, in [DP08], Dwyer and Palmieri show that BL_Λ has cardinality $2^{2^{\aleph_0}}$, although the homogeneous prime spectrum has only one element. Also, the Boolean algebra BA_Λ of complemented classes is trivial, and as a consequence, every smashing localization is trivial [DP08, Cor. 7.5].

Define $I(\Lambda) = \mathrm{Hom}^*_k(\Lambda, k)$ to be the graded vector space dual of Λ; this is also a graded Λ-module. We consider graded Λ-modules as objects of $D(\Lambda)$, concentrated at homological degree zero. The distributive lattice DL_Λ is defined to be the collection of Bousfield classes $\langle X \rangle$ such that $\langle X \rangle = \langle X \wedge X \rangle$. For $D(\Lambda)$, this is a proper sub-poset of BL_Λ, because $\langle I(\Lambda) \wedge I(\Lambda) \rangle = \langle 0 \rangle \neq \langle I(\Lambda) \rangle$ [DP08, Cor. 4.12]. This implies that there is no Noetherian ring that stratifies $D(\Lambda)$, in the sense of [BIK11].

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The picture is very different if we consider a commutative (ungraded) Noetherian ring T and the unbounded derived category $D(T)$. In this context, there is an isomorphism of lattices between BL_T and the lattice of subsets of the prime spectrum [Nee92, HPS97]. The category $D(T)$ is stratified by T [BIK11, Ex. 4.4], and we have that $\mathsf{BA}_T = \mathsf{DL}_T = \mathsf{BL}_T$.

In the Noetherian context, it is the case that every object is Bousfield equivalent to a module. Furthermore, the main results on BL_Λ use Bousfield classes of modules. This prompted Dwyer and Palmieri to ask [DP08, Question 5.8]: for a commutative ring R, is every object in $D(R)$ Bousfield equivalent to an R-module?

The purpose of this paper is to show that in $D(\Lambda)$ the answer is no. In Section 3 we construct an object Tel in $D(\Lambda)$, with homology groups $H_n(\mathsf{Tel}) \cong I(\Lambda)$ for all $n \in \mathbb{Z}$, and use this to prove the following.

Theorem 1. There are objects in $D(\Lambda)$ that are not Bousfield equivalent to a module. Specifically, there are $I(\Lambda)$-acyclics that are not Tel-acyclics, and any such object cannot be Bousfield equivalent to a module.

This appears as Theorem 3.5. We briefly sketch the proof here. First, we show that Tel and $I(\Lambda)$ are not Bousfield equivalent (Lemma 3.3). Then, using a spectral sequence argument, and the fact that Tel has homology $I(\Lambda)$ in each degree (Prop. 3.1), we show that a module is Tel-acyclic if and only if it is $I(\Lambda)$-acyclic (Lemma 3.4). A simple proof by contradiction then implies that not every object is Bousfield equivalent to a module.

The proof is not constructive, and we have been unable to construct such an object. The proof relies on the ring Λ being non-Noetherian, and graded so that each graded piece is a finite-dimensional vector space. Furthermore, the proof uses in a crucial way the fact that $\langle I(\Lambda)\rangle$ is the minimum non-zero Bousfield class in BL_Λ [DP08, Cor. 7.3]. This seems to be a very special property of $D(\Lambda)$. In [Wol12] we investigated non-Noetherian rings similar to Λ, and there we showed in Corollary 4.2.3 that if in the definition of Λ one replaces the field k with the p-local integers $\mathbb{Z}_{(p)}$ for some prime p, then the resulting derived category has no minimum non-zero Bousfield class.

The outline of the paper is as follows. Section 2 includes background on Bousfield classes and Bousfield lattices, which can be defined in any well-generated tensor-triangulated category. Here we also discuss other categories, to put $D(\Lambda)$ in context. In Section 3 we construct the object Tel, and prove the theorem. The proof that $H_n(\mathsf{Tel}) \cong I(\Lambda)$ for all $n \in \mathbb{Z}$ is deferred to Section 4.

2. Review of Bousfield classes and examples

In this section we review the definition and basic properties of Bousfield classes and the Bousfield lattice, and outline some of what is known about the Bousfield lattice in several examples.

Let C be a compactly generated tensor triangulated category [BIK08, §8]; denote the tensor product by $- \wedge -$ and the unit by S. Recall that we defined the *acyclics* of an object X to be all W such that $W \wedge X = 0$, and defined two objects X and Y to be *Bousfield equivalent* if they have the same acyclics. The equivalence class of an object X, under this equivalence relation, is denoted $\langle X\rangle$ and called the *Bousfield class* of X. Most of the following general properties of Bousfield classes were first established by Bousfield [Bou79a, Bou79b] in the context of the stable homotopy category. Further work was done in [Rav84, HPS97, HP99, IK11].

There is a partial ordering on Bousfield classes, given by reverse inclusion. So we say

$$\langle X\rangle \leq \langle Y\rangle \text{ if and only if } W\wedge Y = 0 \implies W \wedge X = 0.$$

The maximum Bousfield class is $\langle S\rangle$ and the minimum is $\langle 0\rangle$. There is a join operation, given by

$$\bigvee_{\alpha} \langle X_\alpha\rangle = \left\langle \coprod_{\alpha} X_\alpha \right\rangle.$$

When C is a well-generated category, there is a set of Bousfield classes [IK11, Thm. 3.1]. All the specific examples of tensor-triangulated categories we will consider in this paper are compactly-generated, and hence well-generated. In any poset with a join and a maximum element, we can define the meet of two elements $\langle X\rangle$ and $\langle Y\rangle$ as the join of (the *set* of) all the lower bounds of $\langle X\rangle$ and $\langle Y\rangle$. The collection of Bousfield classes is then a poset with finite meets and arbitrary joins, i.e. a complete lattice, called the *Bousfield lattice*.

Note that the tensor product gives another operation of Bousfield classes, by

$$\langle X\rangle \wedge \langle Y\rangle := \langle X \wedge Y\rangle.$$

This is a lower bound, but in general not the meet. To address this, we can restrict to the collection DL of classes $\langle X\rangle$ with $\langle X\rangle = \langle X\wedge X\rangle$. One can show that in this sub-poset, the meet of two classes is indeed given by tensoring. Since the tensor product commutes with finite (resp. infinite) coproducts, this meet commutes with finite (resp. infinite) joins, and this sub-poset is a *distributive lattice* (resp. *frame*).

A Bousfield class $\langle X\rangle$ is called *complemented* if there exists a class $\langle X\rangle^c$ such that $\langle X\rangle \wedge \langle X\rangle^c = \langle 0\rangle$ and $\langle X\rangle \vee \langle X\rangle^c = \langle S\rangle$. The collection of complemented classes forms a Boolean algebra, denoted BA, and one always has $\mathsf{BA} \subseteq \mathsf{DL} \subseteq \mathsf{BL}$.

Example 2.1. Iyengar and Krause [IK11] investigate the Bousfield lattice of a tensor-triangulated category that is stratified by the action of a graded Noetherian ring R. This general setting, developed in [BIK08, BIK11], building on [Nee92, BCR97, HPS97] includes the unbounded derived category of a commutative Noetherian ring; the stable module category $\mathrm{StMod}(kG)$ of a finite group, where the characteristic of k divides the order of the group, and then also the homotopy category $K(\mathrm{Inj}\; kG)$ of complexes of injectives; and DG modules over a formal commutative DG algebra with a Noetherian cohomology ring. They show that in such a category the Bousfield lattice is isomorphic to the lattice of subsets of the homogeneous prime spectrum of R, and $\mathsf{BA} = \mathsf{DL} = \mathsf{BL}$.

Example 2.2. We will discuss the case of a commutative Noetherian ring more explicitly. Let R be an (ungraded) commutative Noetherian ring, and $D(R)$ the unbounded derived category of (ungraded) R-modules. Let $\mathsf{Spec}\, R$ be the prime ideal spectrum. Given a prime ideal $\mathfrak{p} \in \mathsf{Spec}\, R$, let $k_\mathfrak{p}$ denote the residue field $(R/\mathfrak{p}R)_\mathfrak{p}$. Define the *support* of an object X in $D(R)$ to be

$$\mathrm{supp}(X) = \{\mathfrak{p} \in \mathsf{Spec}\; R \mid X \otimes^L_R k_\mathfrak{p} \neq 0\}.$$

It is not hard to show that $\mathrm{supp}(k_\mathfrak{p}) = \{\mathfrak{p}\}$. For any objects X and Y in $D(R)$, we have $\langle X\rangle = \langle Y\rangle$ if and only if $\mathrm{supp}(X) = \mathrm{supp}(Y)$ [HPS97, Thm. 6.1.5]. The isomorphism between BL_R and the lattice of subsets of $\mathsf{Spec}\, R$ implies that every object is Bousfield equivalent to a module. Specifically, for any X in $D(R)$ we have

$$\langle X \rangle = \left\langle \bigoplus_{\mathfrak{p} \in \mathrm{supp}(X)} k_{\mathfrak{p}} \right\rangle .$$

Recently Dell'Ambrogio and Stevenson [DS11] have developed a similar notion of support that applies to the derived category $D(T)$ of graded modules over a graded commutative Noetherian ring T. They show that there is an isomorphism between the Bousfield lattice and subsets of the homogeneous prime spectrum. Moreover, this implies that every object in $D(T)$ is Bousfield equivalent to a module.

Example 2.3. In the stable homotopy category, Bousfield [Bou79a] showed that the class of every finite spectrum is in BA, and the class of $H\mathbb{Z}$ is in DL but not in BA. He also showed that the Brown-Comenetz dual I of the sphere has $I \wedge I = 0$, so $\mathsf{DL} \subsetneq \mathsf{BL}$. Hovey and Palmieri [HP99] study finer structure of the Bousfield lattice of this category.

The analog of a module is perhaps an Eilenberg-MacLane spectrum $K(G, 0)$, which has nth homotopy zero for $n \neq 0$. It seems unlikely that every spectrum is Bousfield equivalent to an Eilenberg-Moore spectrum, but we haven't pursued this question.

It is in the context of these examples that we consider the graded ring Λ and the derived category $D(\Lambda)$, defined in the introduction. Neeman [Nee00] considered a similar ring (with denominator generated by x_i^i, for $i \geq 1$, rather than x_i^2) and showed the Bousfield lattice is large although the homogeneous prime spectrum is trivial. Dwyer and Palmieri [DP08] considered a slightly more general ring, with denominator generated by $x_i^{n_i}$ for some fixed $n_i \geq 2$, for each $i \geq 1$. Theorem 1 actually holds in this generality, but we have restricted to the case where $n_i = 2$ for each i, to simplify the proofs. See [Wol12, §6.1.3] for the general case.

In [DP08] the authors require the field k to be countable, but only in order to guarantee that there is a set of Bousfield classes in $D(\Lambda)$: the derived category of a countable ring is a *Brown category* (i.e. Brown representability holds for homology theories, see [HPS97, Thm.9.3.1]), and [DP01] shows that every Brown category has a set of Bousfield classes. However, as discussed above, [IK11, Thm. 3.1] allows us to remove this restriction on k.

The module $I(\Lambda)$ plays a central role in [DP08]. As mentioned in the introduction, $I(\Lambda) \wedge I(\Lambda) = 0$, so $\mathsf{DL}_\Lambda \subsetneq \mathsf{BL}_\Lambda$. This is relevant, because it implies that there is no Noetherian ring that stratifies $D(\Lambda)$. Furthermore, Iyengar and Krause [IK11] have defined a notion of support based on the distributive lattice DL, which makes sense in any well-generated tensor-triangulated category. This generalizes the notion of support defined above, and the support theory used in Example 2.1, but may not be the right thing for derived categories of non-Noetherian rings like Λ.

Corollary 7.3 in [DP08] shows that for any non-zero E in $D(\Lambda)$, we have that $\langle I(\Lambda) \rangle \leq \langle E \rangle$. This implies that BA_Λ is trivial – the only complemented pair is $\langle 0 \rangle$ and $\langle S \rangle$. Indeed, given a complemented pair $\langle X \rangle$ and $\langle X \rangle^c$, one of them must be $\langle 0 \rangle$ or else $\langle 0 \rangle \neq \langle I(\Lambda) \rangle \leq \langle X \rangle \wedge \langle X \rangle^c$.

We mention one final distinction among the Bousfield lattices in these examples. A triangulated subcategory is called *localizing* if it is closed under coproducts.

One can easily check that every Bousfield class is a localizing subcategory. Hovey and Palmieri [HP99, Conj. 9.1] conjecture the converse, in the stable homotopy category. In a category that is stratified by the action of a Noetherian ring, it is indeed the case that every localizing subcategory is a Bousfield class [IK11, Cor. 4.5]. Recently, Greg Stevenson [Ste12], working in the unbounded derived category of a non-Noetherian ring (specifically any absolutely flat ring which is not semi-artinian), exhibited a localizing subcategory that is not a Bousfield class.

3. Tel and modules in $D(\Lambda)$

Now we begin to work towards proving the main theorem. Recall that we defined Λ to be the graded ring

$$\Lambda := \frac{k[x_1, x_2, x_3...]}{(x_1^2, x_2^2, x_3^2...)},$$

where k is a field and $\deg(x_i) = 2^i$.

First we establish some notation. For a Λ-module $M = \oplus_{i\in\mathbb{Z}} M_i$, let $M[n]$ denote the shifted Λ-module with ith piece $(M[n])_i = M_{i-n}$. We grade the chain complexes in $D(\Lambda)$ homologically, so objects in $D(\Lambda)$ are bigraded, and the differential lowers homological grading by one but preserves module gradings. Let Σ denote the suspension in $D(\Lambda)$, and Σ^n the n-fold suspension. This convention is used in order to agree with the definition of $D(\Lambda)$ given in [DP08].

If $x \in \Lambda$ is a homogeneous element of Λ of degree r, then we write $\Lambda \xrightarrow{x} \Lambda[-r]$ for the graded map of degree zero that is multiplication by x. However, for simplicity we will often neglect to include the shift, writing only $\Lambda \xrightarrow{r} \Lambda$. The same is true for maps from a subquotient of Λ.

Let C in $D(\Lambda)$ be represented by the following chain complex.

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \Lambda & \xrightarrow[x_1]{} & \Lambda[-2] & \xrightarrow[x_1x_2]{} & \Lambda[-8] & \xrightarrow[x_2x_3]{} & \Lambda[-20] & \xrightarrow[x_3x_4]{} & \Lambda[-44] & \xrightarrow[x_4x_5]{} & \Lambda[-92] & \xrightarrow[x_5x_6]{} & ..., \\
 & & (0) & & (-1) & & (-2) & & (-3) & & (-4) & & (-5) & &
\end{array}$$

We have indicated the homological degrees below each module. One can check that in homological degree $-n$ we have $\Lambda[4 - 3 \cdot 2^n]$, when $n \geq 1$.

Define $f : C \to \Sigma^2 C$ to be the following chain map.

$$\begin{array}{ccccccccccccccc}
 & & & & 0 & \xrightarrow{0} & \Lambda & \xrightarrow{x_1} & \Lambda[-2] & \xrightarrow{x_1x_2} & \Lambda[-8] & \xrightarrow{x_2x_3} & \Lambda[-20] & \xrightarrow{x_3x_4} & \cdots \\
 & & & & {\scriptstyle 0}\downarrow & & {\scriptstyle x_3}\downarrow & & {\scriptstyle x_1x_4}\downarrow & & {\scriptstyle x_2x_5}\downarrow & & {\scriptstyle x_3x_6}\downarrow & & \\
0 & \xrightarrow[0]{} & \Lambda & \xrightarrow[x_1]{} & \Lambda[-2] & \xrightarrow[x_1x_2]{} & \Lambda[-8] & \xrightarrow[x_2x_3]{} & \Lambda[-20] & \xrightarrow[x_3x_4]{} & \Lambda[-44] & \xrightarrow[x_4x_5]{} & \Lambda[-92] & \xrightarrow[x_5x_6]{} & \cdots \\
 & & (2) & & (1) & & (0) & & (-1) & & (-2) & & (-3) & &
\end{array}$$

For notational simplicity, we will write the map $f : C \to \Sigma^2 C$ as follows.

$$
\begin{array}{ccccccccccccccccccc}
 & & & & 0 & \xrightarrow{0} & \Lambda & \xrightarrow{x_1} & \Lambda & \xrightarrow{x_1x_2} & \Lambda & \xrightarrow{x_2x_3} & \Lambda & \xrightarrow{x_3x_4} & \Lambda & \xrightarrow{x_4x_5} & \Lambda & \longrightarrow & \cdots . \\
 & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle x_3} & & \downarrow{\scriptstyle x_1x_4} & & \downarrow{\scriptstyle x_2x_5} & & \downarrow{\scriptstyle x_3x_6} & & \downarrow{\scriptstyle x_4x_7} & & \downarrow{\scriptstyle x_5x_8} & & \\
0 & \xrightarrow[0]{} & \Lambda & \xrightarrow[x_1]{} & \Lambda & \xrightarrow[x_1x_2]{} & \Lambda & \xrightarrow[x_2x_3]{} & \Lambda & \xrightarrow[x_3x_4]{} & \Lambda & \xrightarrow[x_4x_5]{} & \Lambda & \xrightarrow[x_5x_6]{} & \Lambda & \xrightarrow[x_6x_7]{} & \Lambda & \longrightarrow & \cdots \\
 & & (2) & & (1) & & (0) & & (-1) & & (-2) & & (-3) & & (-4) & & (-5) & &
\end{array}
$$

One can check that with the grading $\deg(x_i) = 2^i$, this is in fact a chain map. Now define Tel to be the telescope

$$\mathsf{Tel} = f^{-1}C = \mathsf{colim}\left(C \xrightarrow{f} \Sigma^2 C \xrightarrow{\Sigma^2 f} \Sigma^4 C \longrightarrow \cdots \right).$$

Recall that, given a self-map $X \xrightarrow{f} X$ in any derived category $D(R)$, the telescope $f^{-1}X$ is defined to be the cofiber of the map $\coprod_{i\geq 0} X_i \xrightarrow{1-f} \coprod_{i\geq 0} X_i$, where $X_i = X$ for all i and the map sends each summand $X_i \to X_i \coprod X_{i+1}$ by $(1-f)(x) = (x, -f(x))$. This is a minimal weak colimit (see e.g. [HPS97, Prop. 2.2.4]), so for all n we have

$$H_n(f^{-1}X) \cong \varinjlim (H_n(X) \xrightarrow{H(f)} H_n(X) \to \cdots).$$

Proposition 3.1. *For all $n \in \mathbb{Z}$, there is a graded Λ-module isomorphism*

$$\tau : H_n(\mathsf{Tel}) \xrightarrow{\sim} (I(\Lambda))[2].$$

We defer the proof to the next section. Next we must extend our definition of $I(-)$ from Λ to mod-Λ to $D(\Lambda)$.

Definition 3.2.
(1) Given a graded Λ-module M, define $I(M) = \mathrm{Hom}_k^*(M, k)$, the graded k-vector space dual. Since Λ is commutative, this has a right Λ-module structure defined by $(f.\sigma)(x) = (\sigma.f)(x) = f(x.\sigma)$ for $f \in I(M)$, $\sigma \in \Lambda$, and $x \in M$.
(2) Let $\mathbb{R}\mathrm{Hom}_k(-,-)$ denote $\mathbb{R}\mathrm{Hom}$ in the derived category of graded k-modules. For any X in $D(\Lambda)$, define $I(X) = \mathbb{R}\mathrm{Hom}_k(X, k)$. Specifically, since k is self-injective, given a chain complex X with Λ-module X^n in homological degree n, $I(X)$ is represented by the chain complex with $\mathrm{Hom}_k^*(X^n, k)$ in degree n. Since this is a Λ-module and the induced differentials are Λ-module maps, we can think of $I(X)$ as an object in $D(\Lambda)$.

Note that if a Λ-module M is *locally finite* (i.e. finite-dimensional in each degree), then $I(I(M)) \cong M$. In particular, $I(I(\Lambda)) \cong \Lambda$.

Lemma 3.3. $\langle \mathsf{Tel} \rangle \neq \langle I(\Lambda) \rangle$.

Proof. Let K be the cofiber of $f : C \to \Sigma^2 C$. We know that K is not zero, because Proposition 3.1 implies that f is not an equivalence. The following are known about C, K, and Tel [Rav92, Prop. 7.2.6(iii)]:

$$\langle C \rangle = \langle K \rangle \vee \langle \mathsf{Tel} \rangle \text{ and } \langle 0 \rangle = \langle K \rangle \wedge \langle \mathsf{Tel} \rangle.$$

Furthermore, [DP08, Cor. 7.3] shows that $\langle I(\Lambda) \rangle \leq \langle X \rangle$ for all nonzero X in $D(\Lambda)$.

Suppose, towards a contradiction, that $\langle \mathsf{Tel} \rangle = \langle I(\Lambda) \rangle$. Then $\langle \mathsf{Tel} \rangle \leq \langle K \rangle$, so $\langle C \rangle = \langle K \rangle \vee \langle \mathsf{Tel} \rangle = \langle K \rangle$. This implies $\langle 0 \rangle = \langle C \rangle \wedge \langle \mathsf{Tel} \rangle$, so $C \wedge \mathsf{Tel} = 0$. This would force $C \wedge I(\Lambda) = 0$.

But we will now show that $C \wedge I(\Lambda) \neq 0$. As remarked in [DP08, Lemma 3.4], tensor-hom adjointness at the module level yields the following for all X and Y in $D(\Lambda)$.

$$\mathbb{R}\mathrm{Hom}_\Lambda(X, \mathbb{R}\mathrm{Hom}_k(Y, k)) = \mathbb{R}\mathrm{Hom}_k(X \wedge Y, k).$$

In particular, setting $Y = \Lambda$ gives $I(X) = \mathbb{R}\mathrm{Hom}_k(X, k) \cong \mathbb{R}\mathrm{Hom}_\Lambda(X, I(\Lambda))$ for all X in $D(\Lambda)$. Using this, we compute

$$\begin{aligned} I(C \wedge I(\Lambda)) &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C \wedge I(\Lambda), I(\Lambda)) \\ &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C, \mathbb{R}\mathrm{Hom}_\Lambda(I(\Lambda), I(\Lambda))) \\ &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C, I(I(\Lambda))) \\ &\cong \mathbb{R}\mathrm{Hom}_\Lambda(C, \Lambda). \end{aligned}$$

We have

$$H_0(\mathbb{R}\mathrm{Hom}_\Lambda(C, \Lambda)) \cong [C, \Lambda]_0.$$

The algebra Λ is self-injective, because Λ is a P-algebra [Mar83, Thm. 13.12], so $[C, \Lambda]_0$ is homotopy classes of degree zero chain maps from C to Λ. There are nontrivial such classes of maps.

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \Lambda & \xrightarrow{x_1} & \Lambda & \xrightarrow{x_1x_2} & \Lambda & \xrightarrow{x_2x_3} & \Lambda & \longrightarrow & \cdots \\ & & {\scriptstyle 0}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 0}\downarrow & & & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \Lambda & \longrightarrow & 0 & \longrightarrow & \cdots & & & & \end{array}$$

Therefore $I(C \wedge I(\Lambda)) \neq 0$, and $C \wedge I(\Lambda) \neq 0$. □

In Section 5 of [DP08], the authors asks if every object is Bousfield equivalent to the direct sum of its homology groups. The last two results show that this is not true. This was also shown recently in [IK11, Rmk. 4.8].

Lemma 3.4. *A Λ-module, considered as an object in $D(\Lambda)$, is Tel-acyclic if and only if it is $I(\Lambda)$-acyclic.*

Proof. Since $\langle I(\Lambda) \rangle$ is minimum among nonzero Bousfield classes, we know $\langle I(\Lambda) \rangle \leq \langle \mathsf{Tel} \rangle$. Thus if a module in $D(\Lambda)$ is Tel-acyclic, it must be $I(\Lambda)$-acyclic. We will show that if M is a module and $M \wedge I(\Lambda) = 0$, then $M \wedge \mathsf{Tel} = 0$.

In [KM95, Thm. 4.7] the authors construct a strongly convergent Eilenberg-Moore spectral sequence in the category of ($\mathbb{Z}$-graded, so unbounded) DG modules over a differential graded algebra. In order to use this spectral sequence, we must temporarily neglect the grading on Λ. More specifically, let $\overline{\Lambda}$ be the same ring as Λ but ungraded, and let $D(\overline{\Lambda})$ be the derived category of ungraded modules over $\overline{\Lambda}$. There is a forgetful functor from chain complexes of Λ-modules to chain complexes of $\overline{\Lambda}$-modules, that takes acyclic complexes to acyclic complexes (and non-acyclic complexes to non-acyclic complexes). Thus let $U : D(\Lambda) \to D(\overline{\Lambda})$ be the induced forgetful functor, and write $U(X) = \overline{X}$. Note that for any object X in $D(\Lambda)$, $H_n(\overline{X}) \cong \overline{H_n(X)}$.

We can consider $\overline{\Lambda}$ as a differential graded algebra concentrated in chain degree zero, and DG modules X and Y over $\overline{\Lambda}$ are just chain complexes of ungraded $\overline{\Lambda}$-modules, thus represent objects in $D(\overline{\Lambda})$. Then the spectral sequence in [KM95, Thm. 4.7] becomes

$$E^2_{p,q} = \bigoplus_{m+n=q} \mathrm{Tor}_p^{\overline{\Lambda}}(H_m(X), H_n(Y)) \Longrightarrow H_{p+q}(X \wedge Y).$$

Given a chain complex Q and a module N, let $Q \otimes N$ be the obvious chain complex. Now suppose M is a module in $D(\Lambda)$ such that $M \wedge I(\Lambda) = 0$ in $D(\Lambda)$. This says that given a projective resolution P of M in $D(\Lambda)$, we have $P \otimes I(\Lambda)$ acyclic, which is true if and only if $\overline{P \otimes I(\Lambda)}$ is acyclic. This is true if and only if $\overline{P} \otimes \overline{I(\Lambda)}$ is acyclic. But P is a projective Λ-module resolution of M if and only if $\overline{P}$ is a projective $\overline{\Lambda}$-module resolution of $\overline{M}$ [NVO04, §2.2], so we get $\overline{M} \wedge \overline{I(\Lambda)} \cong \overline{P} \otimes \overline{I(\Lambda)}$ acyclic in $D(\overline{\Lambda})$.

Letting $X = \overline{M}$ and $Y = \overline{\mathsf{Tel}}$, the spectral sequence E^2 page becomes

$$E^2_{p,q} = \bigoplus_{m+n=q} \mathrm{Tor}_p^{\overline{\Lambda}}(H_m(\overline{M}), H_n(\overline{\mathsf{Tel}})) = \mathrm{Tor}_p^{\overline{\Lambda}}(\overline{M}, \overline{I(\Lambda)}) = H_p(\overline{M} \wedge \overline{I(\Lambda)}).$$

Since $\overline{M} \wedge \overline{I(\Lambda)} = 0$, this collapses to zero and the spectral sequence is strongly convergent, so we must have $H_{p+q}(\overline{M} \wedge \overline{\mathsf{Tel}}) = 0$ for all p and q. By the same argument as above, any projective resolution of Tel in $D(\Lambda)$ giving $M \wedge \mathsf{Tel} \neq 0$ would also give $\overline{M} \wedge \overline{\mathsf{Tel}} \neq 0$, so we can conclude that $M \wedge \mathsf{Tel} = 0$ in $D(\Lambda)$, as desired. □

Theorem 3.5. *In $D(\Lambda)$, there are objects that are not Bousfield equivalent to any module. Specifically, every $I(\Lambda)$-acyclic object that is not* Tel*-acyclic cannot be Bousfield equivalent to a module.*

Proof. Suppose, towards a contradiction, that every object Y in $D(\Lambda)$ is Bousfield equivalent to some module, M_Y. Take X with $X \wedge I(\Lambda) = 0$. Then $M_X \wedge I(\Lambda) = 0$. Using Lemma 3.4, wee see that $M_X \wedge \mathsf{Tel} = 0$, so $X \wedge \mathsf{Tel} = 0$.

This implies that $\langle I(\Lambda)\rangle \geq \langle \mathsf{Tel}\rangle$. Since we already have $\langle I(\Lambda)\rangle \leq \langle \mathsf{Tel}\rangle$, we conclude that $\langle I(\Lambda)\rangle = \langle \mathsf{Tel}\rangle$. This contradicts Lemma 3.3. □

We have shown that $\langle I(\Lambda)\rangle < \langle \mathsf{Tel}\rangle$, so that there are $I(\Lambda)$-acyclics that are not Tel-acyclic. It would of course be nice to construct such an object more explicitly. One natural candidate may be Tel itself. However, the theorem tells us nothing, since Tel is Tel-acyclic. This follows from Corollary 4.12 in [DP08], which shows that $I(\Lambda) \wedge I(\Lambda) = 0$. If we set $X = Y = \mathsf{Tel}$ in the spectral sequence of Lemma 3.4, then $I(\Lambda) \wedge I(\Lambda) = 0$ implies $\mathsf{Tel} \wedge \mathsf{Tel} = 0$. Without Theorem 3.5, it is not clear how to ascertain whether Tel is or is not Bousfield equivalent to a module.

4. Proof of Proposition 3.1

Our goal is to show that for all $n \in \mathbb{Z}$,

$$H_n(\mathsf{Tel}) \cong \mathsf{colim}\left[H_n(C) \longrightarrow H_n(\Sigma^2 C) \longrightarrow \cdots\right] \cong I(\Lambda)[2].$$

For concreteness, we will compute $H_{-2}(\mathsf{Tel})$, and then indicate the general case. We will split the computation into several lemmas.

Because of the shift, we are trying to compute

$$H_{-2}(\mathsf{Tel}) \cong \mathsf{colim}\left[H_{-2}(C) \xrightarrow{x_2x_5} H_{-4}(C) \xrightarrow{x_4x_7} H_{-6}(C) \longrightarrow \cdots\right].$$

We have

$$H_{-2}(C) = \frac{\ker(x_2x_3)}{\operatorname{im}(x_1x_2)}[-8] \cong \frac{(x_2,x_3)}{(x_1x_2)}[-8], \text{ and } H_{-n}(C) \cong \frac{(x_n,x_{n+1})}{(x_{n-1}x_n)}[4-3\cdot 2^n], \text{ for } n \geq 2.$$

Define

$$M_{-2} = \frac{(x_3)}{(x_3)\cap(x_2,x_4,x_5,x_6,\ldots)}[-8],$$

and in general

$$M_{-n} = \frac{(x_{n+1})}{(x_{n+1})\cap(x_n,x_{n+2},x_{n+3},x_{n+4},\ldots)}[4-3\cdot 2^n],$$

and (omitting module shifts and simplifying denominators for readability) consider the collection of maps

$$\frac{(x_{n+1})}{(x_n,x_{n+2},x_{n+3},\ldots)} = M_{-n} \xrightarrow{x_nx_{n+3}} M_{-n-2} = \frac{(x_{n+3})}{(x_{n+2},x_{n+4},x_{n+5},\ldots)}.$$

Lemma 4.1.

$$\begin{aligned} H_{-2}(\mathsf{Tel}) &\cong \mathsf{colim}\left[\frac{(x_2,x_3)}{(x_1x_2)} \xrightarrow{x_2x_5} \frac{(x_4,x_5)}{(x_3x_4)} \xrightarrow{x_4x_7} \frac{(x_6,x_7)}{(x_5x_6)} \xrightarrow{x_6x_9} \cdots\right] \\ &\cong \mathsf{colim}\left[M_{-2} \xrightarrow{x_2x_5} M_{-4} \xrightarrow{x_4x_7} M_{-6} \xrightarrow{x_6x_9} \cdots\right]. \end{aligned}$$

Proof. This uses the universal property of colim. For all $n \geq 2$, we have surjective projection maps

$$\psi_{-n} : H_{-n}(C) \cong \frac{(x_n,x_{n+1})}{(x_{n-1}x_n)} \longrightarrow \frac{(x_{n+1})}{(x_n,x_{n+2},x_{n+3},x_{n+4},\ldots)} = M_{-n}.$$

These maps are compatible with the colimit maps; one can check that the following square commutes for all r.

$$\begin{array}{ccc} \frac{(x_r,x_{r+1})}{(x_{r-1}x_r)} & \xrightarrow{x_rx_{r+3}} & \frac{(x_{r+2},x_{r+3})}{(x_{r+1}x_{r+2})} \\ \downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle proj} \\ \frac{(x_{r+1})}{(x_r,x_{r+2},x_{r+3},\ldots)} & \xrightarrow{x_rx_{r+3}} & \frac{(x_{r+3})}{(x_{r+2},x_{r+4},x_{r+5},\ldots)} \end{array}$$

Thus we get maps $H_{-n}(C) \to \mathsf{colim} M_i$, which induce $\Psi : \mathsf{colim} H_i(C) \to \mathsf{colim} M_i$. We will show that Ψ is surjective and injective.

<u>surjectivity:</u> We will use standard properties of colimits (see e.g. [Mar83, App. 1.2, Prop. 7]). Take $\tilde{x} \in \mathsf{colim} M_i$. So $\tilde{x}$ is represented by $x \in M_{-r}$ for some r. Since ψ_{-r} is surjective, we can pick a $y \in H_{-r}(C)$ such that $\psi_{-r}(y) = x$. By the definition of a colimit, this factors through Ψ. So, letting $\tilde{y}$ be the image of y in $\mathsf{colim} H_i(C)$, we get $\Psi(\tilde{y}) = \tilde{x}$.

<u>injectivity:</u> Suppose $\Psi(\tilde{y}) = 0$. Then $\tilde{y}$ is represented by $y \in H_{-r}(C)$ for some r. We have a commuting diagram

$$\begin{array}{ccc} H_{-r}(C) & \longrightarrow & \mathsf{colim} H_i(C) \\ \downarrow{\scriptstyle \psi_{-r}} & & \downarrow{\scriptstyle \Psi} \\ M_{-r} & \longrightarrow & \mathsf{colim} M_i \end{array}.$$

Therefore $x = \psi_{-r}(y) \in M_{-r}$ maps to zero in $\mathsf{colim} M_i$. This means that either $x = 0$, or x becomes zero eventually in the sequence $M_{-r} \to M_{-r-2} \to M_{-r-4} \to \cdots$. Suppose that x becomes zero at M_{-r-s}, where it could be that $s = 0$. The following square commutes.

$$\begin{array}{ccc} H_{-r}(C) & \longrightarrow & H_{-r-s}(C) \\ \downarrow{\scriptstyle \psi_{-r}} & & \downarrow{\scriptstyle \psi_{-r-s}} \\ M_{-r} & \longrightarrow & M_{-r-s} \end{array}.$$

Since $\psi_{-r}(y) = x$, this implies that the image of y in $H_{-r-s}(C)$, call it z, maps to zero in M_{-r-s}.

If $z = 0$, then we're done – this implies that $\tilde{y} = 0$. So consider the case that $z \neq 0$, but $\psi_{-r-s}(z) = 0$. Now, ψ_{-r-s} is the map

$$H_{-r-s}(C) \cong \frac{(x_{r+s}, x_{r+s+1})}{(x_{r+s-1}x_{r+s})} \longrightarrow \frac{(x_{r+s+1})}{(x_{r+s}, x_{r+s+2}, x_{r+s+3}, x_{r+s+4}, ...)}.$$

Therefore $z \in (x_{r+s}, x_{r+s+2}, x_{r+s+3}, x_{r+s+4}, ...)$. But from $H_{-r-s}(C)$, the maps encountered in $\mathsf{colim} H_i(C)$ are precisely $x_{r+s}, x_{r+s+2}, x_{r+s+3}, x_{r+s+4}, ...$, so we are guaranteed that eventually z will be sent to zero. This implies that $\tilde{y} = 0$, so Ψ is injective. □

Lemma 4.2.

$$\mathsf{colim}\left[M_{-2} \xrightarrow{x_2x_5} M_{-4} \xrightarrow{x_4x_7} M_{-6} \xrightarrow{x_6x_9} \cdots\right]$$

$$\cong \mathsf{colim}\left[I\left(\frac{k[x_1]}{(x_i^2)}\right)[2] \hookrightarrow I\left(\frac{k[x_1,x_2,x_3]}{(x_i^2)}\right)[2] \hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_i^2)}\right)[2] \hookrightarrow \cdots\right].$$

Proof. First consider $M_{-4} = \frac{(x_5)}{(x_4,x_6,x_7,...)}[-44]$. As a Λ-module, this has generator x_5, in degree $-44 + 2^5 = -12$, and top degree element $x_1x_2x_3x_5$, in degree

$$-44 + 2^1 + 2^2 + 2^3 + 2^5 = 2.$$

Let $\overline{x_i}$ denote the dual of x_i; it has degree $-\deg(x_i)$. As a Λ-module, $I\left(\frac{k[x_1,x_2,x_3]}{(x_i^2)}\right)[2]$ is generated by $\overline{x_1x_2x_3}$, in degree $2-2^1-2^2-2^3 = -12$, and has top degree element $\overline{1}$, in degree 2. In fact, we can define a Λ-isomorphism from

$$\frac{(x_5)}{(x_4,x_6,x_7,...)}[-44] \longrightarrow I\left(\frac{k[x_1,x_2,x_3]}{(x_i^2)}\right)[2],$$

by sending $x_5 \mapsto \overline{x_1x_2x_3}$.

Similarly, for all $n \geq 2$, we have Λ-isomorphisms

$$M_{-n} = \frac{(x_{n+1})}{(x_n, x_{n+2}, x_{n+3}, x_{n+4}, ...)}[4 - 3 \cdot 2^n] \longrightarrow I\left(\frac{k[x_1,x_2,...,x_{n-1}]}{(x_i^2)}\right)[2],$$

defined by sending

$$x_{n+1} \mapsto \overline{x_1x_2\cdots x_{n-1}}.$$

The degree of x_{n+1} is $4-3\cdot 2^n+2^{n+1}=4-2^n$, and the degree of $\overline{x_1x_2\cdots x_{n-1}}$ is

$$2-\left(2^1+\cdots+2^{n-1}\right)=2-(2^n-2)=4-2^n.$$

Now, we will show that the maps among the M_i's become inclusions among the duals. First, consider an example.

$$\begin{array}{ccc} M_{-2}=\frac{(x_3)}{(x_2,x_4,x_5,\ldots)}[-8] & \xrightarrow{x_2x_5} & \frac{(x_5)}{(x_4,x_6,x_7,\ldots)}[-44]=M_{-4}\ . \\ \cong\downarrow & & \downarrow\cong \\ I\left(\frac{k[x_1]}{(x_i^2)}\right)[2] & \longrightarrow & I\left(\frac{k[x_1,x_2,x_3]}{(x_i^2)}\right)[2] \end{array}$$

In the bottom left, the generator $\overline{x_1}$ goes up to the generator x_3, then right to $x_2x_3x_5$, which gets sent down to

$$x_2x_3.(\overline{x_1x_2x_3})=\overline{x_1}$$

in the bottom right, and all these maps are degree zero.

In general, we have

$$\begin{array}{ccc} M_{-r}=\frac{(x_{r+1})}{(x_r,x_{r+2},x_{r+3},\ldots)}[4-3\cdot 2^r] & \xrightarrow{x_rx_{r+3}} & \frac{(x_{r+3})}{(x_{r+2},x_{r+4},x_{r+5},\ldots)}[4-3\cdot 2^{r-2}]=M_{-r-2}\ . \\ \cong\downarrow & & \downarrow\cong \\ I\left(\frac{k[x_1,x_2,\ldots,x_{r-1}]}{(x_i^2)}\right)[2] & \longrightarrow & I\left(\frac{k[x_1,x_2,\ldots,x_{r+1}]}{(x_i^2)}\right)[2] \end{array}$$

The generator in the bottom left is $\overline{x_1x_2\cdots x_{r-1}}$, which is sent up to the generator x_{r+1}, then over to $x_rx_{r+1}x_{r+3}=(x_rx_{r+1}).x_{r+3}$. This gets sent down to

$$(x_rx_{r+1}).\overline{x_1x_2\cdots x_{r+1}}=\overline{x_1x_2\cdots x_{r-1}}.$$

This shows that each map becomes the natural degree-zero inclusion under the isomorphisms just described. □

Lemma 4.3.

$$\mathsf{colim}\left[I\left(\frac{k[x_1]}{(x_i^2)}\right)[2]\hookrightarrow I\left(\frac{k[x_1,x_2,x_3]}{(x_i^2)}\right)[2]\hookrightarrow I\left(\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_i^2)}\right)[2]\hookrightarrow\cdots\right]$$

$$\cong I\left(\lim\left[\cdots\to\frac{k[x_1,x_2,x_3,x_4,x_5]}{(x_i^2)}[-2]\xrightarrow{proj}\frac{k[x_1,x_2,x_3]}{(x_i^2)}[-2]\xrightarrow{proj}\frac{k[x_1]}{(x_i^2)}\right][-2]\right)\cong I(\Lambda)[2].$$

Proof. Let $V_j=\frac{k[x_1,x_2,\ldots,x_j]}{(x_i^2)}[-2]$; it's clear that $I(V_j)=I\left(\frac{k[x_1,x_2,\ldots,x_j]}{(x_i^2)}\right)[2]$. Since these are locally finite, we have $I(I(V_j))\cong V_j$ for all j. The definition of a sequential colimit gives a certain exact sequence

$$\coprod I(V_j)\xrightarrow{G}\coprod I(V_j)\longrightarrow(\mathsf{colim}I(V_j))\longrightarrow 0,$$

which dualizes to an exact sequence

$$0\longrightarrow I(\mathsf{colim}I(V_j))\longrightarrow\prod I(I(V_j))\xrightarrow{I(G)}\prod I(I(V_j)).$$

One can check that in fact $I(G)$ is the map used in the definition of the sequential limit, so we have

$$\lim V_j\cong I(\mathsf{colim}I(V_j)).$$

Since $\lim V_j \cong \Lambda$, this shows that $\mathsf{colim} I(V_j)$ is the thing that dualizes to Λ. In other words $\mathsf{colim} I(V_j) \cong I(\Lambda)$. □

Proof of Proposition 3.1. Combining the three previous lemmas, we have an isomorphism $\tau : H_{-2}(\mathsf{Tel}) \xrightarrow{\sim} I(\Lambda)[2]$. Because the map $f : C \to \Sigma^2 C$ has degree two, and sequential colimits are determined by their long-term behavior, it's easy to see that $H_i(\mathsf{Tel}) \cong H_{-2}(\mathsf{Tel}) \cong I(\Lambda)[2]$ for all even i.

Additionally, a computation of $H_{-3}(\mathsf{Tel})$, for example, would proceed as above, but with all indices incremented/decremented by one. The result is the same: $H_{-3}(\mathsf{Tel}) \cong H_i(\mathsf{Tel}) \cong I(\Lambda)$ for all odd i, and we have checked that this gives the same module shift, two. Therefore, τ induces $H_i(\mathsf{Tel}) \cong I(\Lambda)[2]$ for all i. □

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