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The majority of my conventional research has been in tensor triangulated category theory, at the intersection of algebraic topology, derived categories, and homological algebra. In the last year I have also been studying topological data analysis. My working hypothesis is: *mathematicians create perspectives, and discover consequences.* My research process centers on a conscious effort to balance the visionary and the technical.

My most recent research paper [Wol15a] reposed and answered a question from classical stable homotopy theory, the telescope conjecture, in localized categories of spectra. These results land squarely in topology. Before that, I published a paper [Wol14] on the structure of derived categories of some non-Noetherian rings – results that belong to homological algebra. The common theme of these two different papers is an appeal to bigger-picture methods of tensor triangulated category theory, and more specifically an investigation of a certain lattice (the Bousfield lattice) that can be extracted from such categories. In Sections 1.1 and 1.2 of this document, I will give more details and will describe my contributions, my current lines of research, and some opportunities for undergraduate research.

Topological data analysis is a new and exciting approach to applying topology to study the “shape of data”. Recently category theory has been brought to bear on the more theoretical and structural questions of TDA. Furthermore, interesting data sets can be subjected to this analysis, and no one really knows what we’ll find; theory and application are leap-frogging each other and there are potentials for student involvement. More details are in Section 1.3.

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However, the real targets of my mathematical inquiry – the questions that I confront with passion and that keep me up at night – center on *understanding the experience of doing mathematics.* What is it like, and how do we do it, when we learn, understand, create/discover, communicate, or teach math? I have many projects, some quite quirky, flowing from this sustained inquiry. I will describe the main directions briefly here, with much more detail in Section 2.

The User’s Guide project. A user’s guide – at the same time humanistic and technical – is written to accompany a published research article, providing further exposition and context for the results. I have started an informal journal, *Enchiridion*, in which authors write user’s guides to their own published research papers, and work collaboratively to group-peer-review each others’ submissions. The goal is to make research mathematics more accessible, to explore unconventional expositional styles, and to augment rigor with humanistic meta-data.

The project website, where the first issue is available, is `mathusersguides.com`. An introduction to the project appeared in [MMW$^+$15]; the second issue and a meta-analysis of the project are in progress. See Section 2.1.

Mathematical phenomenology. What does it mean to understand mathematics? What does it feel like? What are characteristics of the experience of having an advanced understanding of a mathematical subject? In collaboration with philosopher of science Alexandra Van-Quynh, and using methodology from Phenomenology, I have conducted a focus group, survey, and interviews with research mathematicians. Our first paper is in progress, but a preview PDF is available on my website[1].

To contrast the experience of experts and novices, at Lawrence I conducted an IRB-approved study that videotaped students working together on a group theory problem set, and followed up with one-on-one interviews. The analysis of these videos is in progress. See Section 2.2.

Math-art collaborations. I have a large portfolio of collaborations with artists – exhibited work, performance, video, song, and published papers – that address the mathematical experience. Recent projects include a student musical performance of student poems based on

[1]See `tinyurl.com/oj41472`

mathematicians' reflections, and drawings and calculations done while trekking in the Indian Himalaya. For more details, see `forthelukeofmath.com` or Section 2.3.

Contemplative pedagogy. As I discuss in my teaching statement, for several years I have been involved in the contemplative education movement, for example using short mindfulness activities to start class. This year I created a collaborative wiki site[2] to build a database of pedagogical techniques that apply specifically to college-level mathematics courses. Along with Justin Brody, at Goucher College, I am organizing a contributed paper session at the 2016 Joint Meetings in Seattle, titled *Contemplative Pedagogy and Mathematics.*

Exposition and outreach. My interest in understanding the mathematical experience is partially motivated by my desire to share this experience with others. Many of my math-art projects, for example, have been designed to reveal the mathematical experience to non-mathematicians. As documented in my CV, since 2009 I have been working to find unique and unconventional ways of sharing mathematics. For example, the *Gardens of Infinity* project, up at `gardensofinfinity.com`, is a recent collaboration with an interaction designer and programmer. The site is a web-based interactive experience aimed at the average web surfer with a curiosity about infinity. See Section 2.4 for more details and examples.

1. Conventional research

I will first describe my work in homological algebra, relating to [Wol14], then my work in stable homotopy theory that resulted in [Wol15a], and finally my recent involvement with topological data analysis.

Each subsection provides some background, discusses my contributions, and discusses future work. Sections 1.1.4 and 1.3.3 describe some opportunities for undergraduate involvement.

1.1. Derived categories and the Bousfield lattice.

1.1.1. *Background.* Let T be a tensor triangulated category with tensor (i.e. smash) product denoted $\wedge$ and unit $\mathbb{1}$. For example, one can take T to be the (p-local) stable homotopy category $\mathcal{S}$ of spectra, or the derived category $D(R)$ of unbounded chain complexes of R-modules, for a commutative ring R. For an object X of T, the *Bousfield class* of X is defined to be

$$\langle X \rangle = \{W \text{ in } \mathsf{T} \mid W \wedge X = 0\}.$$

It was recently shown by Iyengar and Krause [IK13, Thm. 3.1] that every well generated tensor triangulated category has a set (rather than a proper class) of Bousfield classes. A relatively weak condition, most main examples of tensor triangulated categories are known to be well generated; for example this is the case with $\mathcal{S}$ and $D(R)$. This set of Bousfield classes is called the *Bousfield lattice* $\mathsf{BL}(\mathsf{T})$ and has a lattice structure which I will now briefly describe. Refer to [HP99] or [Wol14] for more details.

The partial ordering is given by reverse inclusion: we say $\langle X \rangle \leq \langle Y \rangle$ when $W \wedge Y = 0 \implies W \wedge X = 0$. Then $\langle 0 \rangle$ is the minimum and $\langle \mathbb{1} \rangle$ is the maximum class. The join of any set of classes is $\bigvee_{i \in I} \langle X_i \rangle = \langle \coprod_{i \in I} X_i \rangle$, and the meet is defined to be the join of all lower bounds.

The smash product induces an operation on Bousfield classes, where $\langle X \rangle \wedge \langle Y \rangle = \langle X \wedge Y \rangle$. This is a lower bound, but in general not the meet. However, if we restrict to the subposet $\mathsf{DL} = \{\langle W \rangle \mid \langle W \wedge W \rangle = \langle W \rangle\}$, then the meet and smash agree. Since coproducts distribute across the smash product, DL is a distributive lattice.

We say a class $\langle X \rangle$ is *complemented* if there exists a class $\langle X^c \rangle$ such that $\langle X \rangle \wedge \langle X^c \rangle = \langle 0 \rangle$ and $\langle X \rangle \vee \langle X^c \rangle = \langle \mathbb{1} \rangle$. The collection of complemented classes is denoted BA. For example, every smashing localization (see Subsection 1.2.1) $L : \mathsf{T} \to \mathsf{T}$ gives a pair of complemented classes. Every complemented class is also in DL, so BA is a Boolean algebra and we have subposet inclusions $\mathsf{BA} \subseteq \mathsf{DL} \subseteq \mathsf{BL}$.

[2]See `contemplativemathematicspedagogy.wordpress.com`

The Bousfield lattice was first introduced by Bousfield [Bou79a], for the category of spectra, and subsequently studied there and in categories arising in algebra and representation theory; see e.g. [Rav84, HPS97, HP99, IK13].

The Bousfield lattice of some categories is basically completely understood. When R is a commutative Noetherian ring, the Bousfield lattice of $D(R)$ is isomorphic to the lattice of subsets of the prime spectrum $\mathsf{Spec}\,R$, ordered by inclusion [Nee92]. This result has been generalized to stratified categories [BIK11a], such as the stable module category of a finite group [BIK11b]. Here the Bousfield lattice is isomorphic to the lattice of subsets of the cohomology ring.

For non-Noetherian rings the picture is less clear. Fix $n_i \geq 0$ and a countable field k, and consider the graded-commutative ring

$$\Lambda_k = \frac{k[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)},$$

with $deg(x_i) = 2^i$. The objects in $D(\Lambda_k)$ are then bigraded. Neeman [Nee00] first considered the Bousfield lattice of such a ring. Dwyer and Palmieri [DP08] show that, although the prime spectrum is trivial, the cardinality of BL is exactly $2^{2^{\aleph_0}}$. Also BA is trivial, and there are objects of arbitrary nilpotence height. Thus not only does the non-Noetherian case differ significantly from the Noetherian one, but it displays similarities with the topological case.

1.1.2. *My contributions.* My thesis work, subsequently written up in [Wol14], looked at the Bousfield lattice of derived categories of several non-Noetherian rings. In particular, fixing a prime p, I considered the graded rings

$$\Lambda_R = \frac{R[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}, \text{ for } R = \mathbb{F}_p, \mathbb{Q}, \text{ and } \mathbb{Z}_{(p)}.$$

I developed a theory for Bousfield lattices of localizing subcategories (i.e. those triangulated subcategories closed under taking coproducts) that are proper; since these don't contain the tensor unit, the relationship between BA, DL, and BL is more subtle. Write $\mathsf{BL}(R)$ as shorthand for $\mathsf{BL}(D(R))$, etc. There are projection and inclusion maps $g : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{F}_p}$ and $h : \Lambda_{\mathbb{Z}_{(p)}} \to \Lambda_{\mathbb{Q}}$, and I showed these induce lattice maps $g_\bullet : \mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}}) \rightleftarrows \mathsf{BL}(\Lambda_{\mathbb{F}_p}) : g^\bullet$ and $h_\bullet : \mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}}) \rightleftarrows \mathsf{BL}(\Lambda_{\mathbb{Q}}) : h^\bullet$.

One main result was the following lattice isomorphisms. Here $\mathsf{loc}(X)$ denotes the smallest localizing subcategory containing X.

Corollaries 5.17 and 5.18 [Wol14]: *The functors $g_\bullet$ and $h_\bullet$ induce lattice isomorphisms*

$$\mathsf{BL}(\Lambda_{\mathbb{Z}_{(p)}}) \cong \mathsf{BL}(\Lambda_{\mathbb{F}_p}) \times \mathsf{BL}(\mathsf{loc}(h^\bullet \Lambda_{\mathbb{Q}})),$$
$$\mathsf{DL}(\Lambda_{\mathbb{Z}_{(p)}}) \cong \mathsf{DL}(\Lambda_{\mathbb{F}_p}) \times \mathsf{DL}(\mathsf{loc}(h^\bullet \Lambda_{\mathbb{Q}})),$$
$$\mathsf{BA}(\Lambda_{\mathbb{Z}_{(p)}}) \cong \mathsf{BA}(\Lambda_{\mathbb{F}_p}) \times \mathsf{BA}(\mathsf{loc}(h^\bullet \Lambda_{\mathbb{Q}})).$$

More generally, [Wol14] considers a ring map $f : A \to B$ between two commutative rings. Extension of scalars and the forgetful functor induce well-behaved adjoint functors $f_\bullet : D(A) \rightleftarrows D(B) : f^\bullet$, and I showed these induce lattice maps $f_\bullet : \mathsf{BL}(A) \rightleftarrows \mathsf{BL}(B) : f^\bullet$. I was able to prove a range of results in this general setting.

In 2013, I shifted from homological algebra to stable homotopy theory. If $L_Z : \mathcal{S} \to \mathcal{S}$ is a homological localization functor on the category $\mathcal{S}$ of p-local spectra, annihilating the Bousfield class $\langle Z \rangle$ of Z_*-acyclics, then the essential image of L_Z has the structure of a well generated tensor triangulated category. This category is called the Z-local category.

The paper [Wol15a] calculates the Bousfield lattice of several local categories of spectra. I give an upper bound, $2^{2^{\aleph_0}}$, on the cardinality of such lattices. Then I show that the $K(n)$-local, $H\mathbb{F}_p$-local, and IS^0-local categories all have two-element Bousfield lattices, where IS^0 is the Brown-Comenetz dual of the sphere. Jon Beardsley has calculated the Bousfield lattice of the harmonic (i.e. $\vee_i K(i)$-local) category to be isomorphic to the power set of $\mathbb{N}$; I show that one can realize this lattice as an inverse limit of the Bousfield lattices of $E(n)$-local categories, as n

ranges over $\mathbb{N}$. Finally, I give a lower bound, $2^{\aleph_0}$, on the cardinality of the Bousfield lattice of the BP-local category.

1.1.3. *Future directions.* Javier Gutiérrez, at the Radboud Universiteit Nijmegen, and I have started a project aimed at understanding the Bousfield lattice of derived categories of R-module spectra, as in [EKMM97]. Given an S-algebra R, the derived category of R-modules $\mathcal{D}(R)$ is a well generated tensor triangulated category. We believe the general theory developed in my thesis and [Wol14], studying the lattice maps $\mathsf{BL}(D(A)) \rightleftarrows \mathsf{BL}(D(B))$ induced by a commutative ring map $f : A \to B$, can be applied here. If $f : R \to T$ is a map of S-algebras, there should be induced lattice maps $\mathsf{BL}(\mathcal{D}(R)) \rightleftarrows \mathsf{BL}(\mathcal{D}(T))$.

One direction to take this, is to let $R = S^0$ be the sphere spectrum, and $T = HA$ be an Eilenberg-MacLane spectrum for a commutative ring A. Then $\mathcal{D}(S^0)$ is the entire category of spectra $\mathcal{S}$, and $\mathcal{D}(HA)$ is equivalent to the algebraic derived category $D(A)$. This will give useful lattice maps between $\mathsf{BL}(\mathcal{S})$ and $\mathsf{BL}(D(A))$, that is, between topology and algebra. Choosing different rings A may help understand $\mathsf{BL}(\mathcal{S})$.

1.1.4. *Opportunities for undergraduate research.* Although category theory is an incredibly elegant field, probing the essence of mathematical structure, it is tough stuff for an undergraduate. However, my research extracts a lattice, the Bousfield lattice, from various categories, and lattice theory is very approachable. Basic results on the distributive lattice DL and Boolean algebra BA in Bousfield lattices have been fleshed out in the past decades, starting with Bousfield himself. But many lattice theory ideas have yet to be applied to Bousfield lattices. The Bousfield lattice of spectra seems to be as complicated a lattice as one could ever hope for, and is very far from understood.

I envision several undergraduate research projects, starting from a lattice theory text such as [Bir79] or [DP02]. Even notions like DL and BA are accessible, computable, and fun. Slightly more advanced lattice theory – chains, idempotent elements, complementation operations – are also accessible, and furthermore are applicable and relevant to current research on Bousfield lattices. While I will need to provide information about various categories as a sort of black box, the students will experience the process of developing a background in classical material, using it to create a new perspective on an area of current research, asking new questions and then seeing what results.

1.2. **Localizing subcategories, telescope conjectures, and localized spectra.**

1.2.1. *Background.* A full subcategory of a triangulated category is *localizing* if it is closed under forming triangles and taking coproducts. A central question of triangulated category theory is to understand or classify localizing subcategories. In many cases, such as the p-local stable homotopy category $\mathcal{S}$, it is not even known if there is a set (rather than a proper class) of localizing subcategories. It is suspected the answer may depend on large cardinal axioms of set theory. Every Bousfield class is a localizing subcategory, and another important question is when the converse is true as well. Greg Stevenson recently gave the first example of a category with a localizing subcategory that is not a Bousfield class [Ste14].

Another important question in tensor triangulated category theory, going back to Bousfield [Bou79b], is to classify smashing localization functors. A localization functor $L : \mathsf{T} \to \mathsf{T}$ on a tensor triangulated category T is, loosely, a useful idempotent functor. A localization functor is *smashing* if it commutes with coproducts. A localizing subcategory is called smashing if it is the kernel of a smashing localization functor.

These questions are at opposite ends, in the following sense. Every smashing localization yields a pair of complemented Bousfield classes, so we have the following chain of inclusions of collections of localizing subcategories.

$$\{\text{smashing localizing subcategories}\} \subseteq \mathsf{BA} \subseteq \mathsf{DL} \subseteq \mathsf{BL} \subseteq \{\text{all localizing subcategories}\}$$

The telescope conjecture, first stated by Ravenel [Rav84, Conj. 10.5], is a famous claim about two classes of localization functors, L_n^f and L_n, in the p-local stable homotopy category of spectra. For all $n \geq 0$, L_n^f and L_n are smashing localizations, and there is a natural map $L_n^f \to L_n$. There are several versions of the telescope conjecture, but it essentially claims that L_n^f and L_n are the same. The conjecture is known to hold for $n = 0$ and $n = 1$. A valiant but unsuccessful effort at a counterexample, for $n \geq 2$, was undertaken by Mahowald, Ravenel, and Shick, as outlined in [MRS01]. Since then little progress has been made, and the original conjecture remains open.

A generalization of the telescope conjecture can be stated for spectra, as well as other triangulated categories. Localization away from a finite spectrum, i.e. a compact object of the category, always yields a smashing localization functor (see e.g. [Mil92] or [HPS97, Thm. 3.3.3]). This is called finite localization. The Generalized Smashing Conjecture (GSC) is that every smashing localization arises in this way. If true, then every smashing localization is determined by its compact acyclics; if the GSC holds in spectra, then so must the original telescope conjecture.

The GSC, essentially stated for spectra decades ago by Bousfield [Bou79b, Conj. 3.4], has been formulated in many other triangulated categories, in many cases labeled as the telescope conjecture, and in many cases proven to hold. Neeman [Nee92] made the conjecture for the derived category $D(R)$ of a commutative ring R, and showed it holds when the ring is Noetherian. On the other hand, Keller [Kel94] gave an example of a non-Noetherian ring for which the GSC fails. Benson, Iyengar, and Krause have shown that the GSC holds in a stratified category [BIK11a], such as the stable module category of a finite group [BIK11b].

1.2.2. *My contributions.* In [Wol15a], I reformulate the telescope conjecture, and GSC, in categories of local spectra. That is, for a localization $L : \mathcal{S} \to \mathcal{S}$, the essential image of L has the structure of a tensor triangulated category, denoted $\mathcal{L}$, which I consider.

One of the main results is the following. I weaken the assumptions for finite localization, and show that in many categories, localization away from any set of strongly dualizable objects yields a smashing localization. (An object X is *strongly dualizable* if $F(X,Y) \cong F(X,\mathbb{1}) \wedge Y$, where $\mathbb{1}$ is the tensor unit and $F(-,-)$ the function object bifunctor.) Specifically, I prove the following.

Theorem A [Wol15a]**:** *Let T be a well generated tensor triangulated category such that $\mathbb{1}$ is strongly dualizable and $\mathsf{loc}(\mathbb{1}) = \mathsf{T}$. Let $A = \{B_\alpha\}$ be a (possibly infinite) set of strongly dualizable objects. Then there exists a smashing localization functor $L : \mathsf{T} \to \mathsf{T}$ with Ker $L = \mathsf{loc}(A)$.*

This leads to the Strongly Dualizable Generalized Smashing Conjecture (SDGSC): Every smashing localization is localization away from a set of strongly dualizable objects.

I give several examples of categories where the GSC fails, but the SDGSC holds. To do this, one must classify smashing localizations on the local category.

Theorem C [Wol15a]**:** *In the harmonic category, the GSC fails but the SDGSC holds. Likewise in the $H\mathbb{F}_p$-local and IS^0-local categories, where IS^0 is the Brown-Comenetz dual of the sphere spectrum. In the BP-local category the GSC fails but the SDGSC may hold. In the $E(n)$-local and $K(n)$-local categories the GSC and SDGSC both hold.*

Furthermore, I define localization functors l_n^f and l_n on $\mathcal{L}$ that are localized versions of L_n^f and L_n. The localized telescope conjecture (LTC), basically, is that l_n^f and l_n are isomorphic. In fact, I give three versions of the LTC, and implications between them. Then, examining specific examples of localized categories of spectra, I conclude the following.

Theorem B [Wol15a]**:** *All versions of the localized telescope conjecture, $\mathsf{LTC1_i}$, $\mathsf{LTC2_i}$, and $\mathsf{LTC3_i}$ hold for all $i \geq 0$, in the harmonic, $K(n)$-local, $H\mathbb{F}_p$-local, BP-local, and IS^0-local categories.*

Finally, I show that in the IS^0-local category there exists a localizing subcategory that is not a Bousfield class. This gives strong evidence to suggest the same is true in the full category of spectra as well.

1.2.3. *Future directions.* Several questions remain from the work in [Wol15a]. Question 3.15 therein asks if l_n is always smashing, as L_n is always smashing on $\mathcal{S}$. Furthermore, I have some ideas for calculating smashing localizations in the BP-local and $F(n)$-local categories. A classification of the localizing subcategories of the harmonic category seems within reach, now that the Bousfield lattice is known to be $2^{\mathbb{N}}$, and I better understand the relationship between the harmonic and $E(n)$-local categories.

1.3. **Topological data analysis.**

1.3.1. *Background.* Imagine a data set X, in the form of a point cloud in some high-dimensional parameter (metric) space. Choose some $\kappa > 0$, and center balls of radius κ at each point in the cloud. The result is a topological space (X, κ). Now imagine continuously or discretely changing κ, and keeping track of all the (X, κ). To reduce the computational complexity of (X, κ), we can replace it with a simplicial complex $V(X, \kappa)$ called the Vietoris-Rips complex, in such a way that for $\kappa_1 < \kappa_2$, we get an inclusion $V(X, \kappa_1) \subseteq V(X, \kappa_2)$. Now we can "do topology" on these spaces, by taking homology $H_i(-)$ for each $i \geq 0$. By varying κ and studying the changes in $H_i(V(X, \kappa))$, we extract information about the original data set. This is called persistent homology [EMD12].

Persistent homology is one – the most common – technique in the decade-old field of topological data analysis. This analysis has been performed – implemented in new software like Javaplex, Perseus, and DIPHA – on interesting data sets, with interesting results. A recent survey [NOMPUT+15] lists applications to breast cancer [NLC11], viral evolution [CCR13], natural images [CIDSZ08], contagion spread on networks [TKH+15], structure of amorphous materials [NHH+15], and collective behavior in biology [CTLZTH14].

The theory of persistent homology has also been developing fast, and recently category theory has been brought more clearly into the picture. From the categorical perspective, for example, the output of persistent homology is an $(\mathbb{R}, \leq)$-indexed diagram in the category of vector spaces. Different choices of sequences of κ yield different diagrams. Going further, a recent paper [BS14] constructs an abelian category of ϵ-interleavings, which are, in a sense, ways of measuring the distance between diagrams coming from different choices of κ sequences.

1.3.2. *My contributions, and future directions.* For a year and a half, I have been slowly catching up and keeping up with the TDA literature, attending online seminars run by the Algebraic Topology Research Network based out of the IMA at the University of Minnesota. In summer 2014, I ran a 4-day workshop and taught the basics of Rips complexes and persistent homology, albeit at a basic level and geared toward non-mathematicians. In winter 2015, I joined the AATRN research team on Category Theory, Sheaf Theory and Applications, and the group of us – spread around the country – are teaching each other and moving towards formulating good research questions. The majority of team members are approaching TDA from the applied mathematics side, and my familiarity and comfort with category theory is an asset. I'm confident that in the near future we will coalesce around good lines of inquiry.

Topological data analysis is exciting because it offers opportunities for both theoretical and practical advances. In addition to studying papers and trying to bring category theory to the theoretical table, I've been teaching myself how to actually perform the analysis on real data sets. I've been running and becoming comfortable with the TDA software Javaplex, running it inside Processing. Processing is a language specifically designed with visualization, and in fact visual art, in mind. Using Javaplex with Processing has taken lots of self-study, but offers much potential for bringing data sets and TDA to life.

1.3.3. *Opportunities for undergraduate research.* I believe there are significant opportunities for student involvement. The basics of TDA can be taught at the undergraduate level. In fact, in December 2015, I plan to co-teach a short course on the basics of TDA at Lawrence, assuming no prerequisites. Since the field is new, a clever student familiar with topology and willing to learn some category theory can reach the frontier of our theoretical knowledge. Furthermore, a student interested in programming or Big Data could, with help, certainly run the persistent homology algorithms on his or her favorite data set.

2. Projects and research on the mathematical experience

2.1. **The User's Guide Project.** Have you ever read a math paper and wondered, how did they come up with this? Why? What's really going on here? What's the best way to think about this?

A user's guide – at the same time humanistic and technical – is written to accompany a published or soon-to-be-published research article, providing further exposition and context for the results. I created an informal journal *Enchiridion*, which brings together five mathematicians to write user's guides on their own papers, and then to work closely together to collaboratively group-edit and peer-review a compilation. Our user's guides are composed of four topics:

> Topic 1: Key insights and organizing principles – What is the conceptual essence of the paper? What's really going on?
>
> Topic 2: Conceptual metaphors and mental imagery –What mental imagery do you have when you think about these results? What conceptual metaphors do you use? How should we think about it?
>
> Topic 3: Story of the development – Where and when did these ideas and results actually arise? To complement those straight logical lines in your proofs, tell us the story of how you actually went from A to B.
>
> Topic 4: Colloquial summary – How would you explain it to a non-mathematician? Additionally, what about this subject is so exciting and meaningful to you?

As I wrote in the published announcement [MMW+15], "One can think of these user's guides as meta-data for the ideas and results in the source paper, attaching a bit more humanity to the objective representations and reasoning. Including this information closes the gap between the practice of mathematics and the artifacts of that practice."

The first issue came out October 2015, and is available at `mathusersguides.com`. I have five new writers working on Volume 2, with finalized topics rolling out throughout the year and a full compilation to be completed by October 2016. The writers, with my help, are responsible for reviewing and editing each others' guides, and this has been generating interesting meta-discussions about mathematical exposition. The Journal of Humanistic Mathematics has asked us to write up an analysis of the project, to be submitted by the end of 2016. My intention is that *Enchiridion* will publish issues annually, each issue bringing together five mathematicians from a common subfield.

This project has grown out of several earlier projects. As a grad student, in 2008 and 2010 I organized and ran a reading group/seminar titled *The Mathematical Experience*, which brought together undergrads, grad students, faculty, and even some staff, to read metamathematical writings and to create an environment to discuss the experiential side of mathematics. This is something I would be thrilled to revive at a new school. My interest in augmenting rigor with experiential meta-data also made its way into my PhD thesis, in which every chapter of rigorous mathematics was concluded with an Experiential Context section.

Additionally, for two years during and after grad school, I documented recurring experiences of doing research mathematics in the *Flavors and Seasons* blog[3]. A *flavor* lasts for a few minutes or hours, for example "working through a proof", whereas a *season*, like "pulling together and writing up", lasts for weeks or months. By answering a fixed set of questions, I would document the cognitive, meta-cognitive, emotional, and logistical aspects of these experiences. In 2013

[3]See `flavorsandseasons.wordpress.com`

I wrote a paper about this blog [Wol13a], making the case that these experiences are shared among our community, and that self-reflection makes for better mathematicians.

2.2. **Mathematical phenomenology.** What does it mean to understand mathematics? What does it feel like? What are characteristics of the experience of having an advanced understanding of a mathematical subject?

One common approach to humanistic questions such as these is qualitative: probing anecdotes from the wise elders of mathematics, as in [Har92, Bye10, Ano, TT, WT10]. Another is quantitative: analyzing statistics from math education studies, for example in comparing the development of procedural versus conceptual understanding in primary and secondary school [RJS98, RJSA01].

I have two ongoing projects that I will discuss here, both taking a middle road between qualitative and quantitative: first, a collaboration with a philosopher of mathematics in which we interviewed professional mathematicians; and second, a gestural study that filmed Lawrence students working together.

2.2.1. *Elicitation interviews.* Since 2013 I have been working on a collaboration with philosopher of mathematics Alexandra Van-Quynh, using the methodology of the *elicitation interview* and its analysis. This methodology from Phenomenology, pioneered by Pierre Vermersch [PV94] in the 1990s and further developed by Bitbol, Petitmengin, Maurel, and many others, aims to bracket and suspend present experience in order to (re)evoke a specific past experience and explore its microdynamics. It has been used to investigate a wide range of experiences – for example seizures [PBN06], visual perception [PRCCT13], listening to music [Po09], intuition [Pet99], and mathematical intuition [AVQ15]. Phenomenology sees reality as comprised of lived experience, and every event as an interconnected unfolding of a perceiver and a perceived. Our elicitation interviews provided us with detailed descriptions of the lived experience of mathematical understanding, from which we are working to extract a generic intersubjective structure.

We specifically chose to look at the experience of understanding groups. Groups are basic objects that all research mathematicians know about; they are a representative example of modern abstract mathematics; there is rarely canonical visual imagery associated to groups. With visual faculties handicapped in this way, our hope in investigating the experience of groups was that non-visual modes of reasoning and perception would be amplified.

In 2013 we conducted a focus group to help develop a list of survey questions about the experience of working with groups. This survey was sent out to practicing research mathematicians, and from the 30 responses we moved to focus on the question of understanding. In 2014 Alexandra, who is trained in the elicitation interview technique, conducted four 1-2 hour interviews with a subset of survey respondents. Questions focused first on the experience of a group that the mathematician reported understanding very well, and then on the experience of a group that was reported as not understood or not fully understood.

The analysis of the interviews is still in progress, but there is a short summary PDF available on my website that summarizes our findings[4]. We expect the interviews will result in at least two published articles.

To discuss one example, we found a common trend in the development of mental imagery. When a mathematician is led to evoke the experience of working with a group that they do not understand (or understand in a basic way), whatever imagery is present is specific, canonical, and symbolic. They describe imagery based on the definition, or only see the standard symbol used to represent the group. They clearly struggle to visualize the group – "To not understand, is to not have a clear image of its elements"; "You don't have a good picture of it"; "you don't see them all, there's too many".

On the other hand, we found much evidence that when a mathematician is led to evoke the experience of working with a group that he or she understands well, the imagery becomes

[4] See `tinyurl.com/oj4l472`

more streamlined, efficient, and intentionally fuzzy. We heard: "I was seeing sort of a pentagon/hexagon... something with some sides but not too many sides, because my mind cannot see too many sides, not too few sides because then it's too particular"; "It's really an archetype. It's an archetype of an element. It's not an element in its own"; "It's like there's this framework"; "They don't need to be precise"; "So when the number gets bigger, I just let the picture become a little fuzzier, and I pretend that I'm still imagining it just as well".

It seems that having too many details adds unnecessary cognitive weight to the image, and slows down reasoning. The mathematician who understands a group will leave out those unnecessary details in his or her images, in a gradual streamlining process. This push to streamline keeps the images primitive and fuzzy, while remaining incredibly useful. The paradox of images that are primitive yet useful is that they are useful because they are primitive.

2.2.2. *Undergraduates working together on group theory.* To contrast the experience of experts and novices, I am conducting an IRB-approved study with Lawrence undergraduates. What does understanding mean to an undergraduate, and how is this different than the experience of an expert?

In winter 2015 I was teaching a course on group theory. Two specific lectures, in which I taught about the symmetric group and cycle notation, were filmed. Five of my students volunteered to participate in the study. A month after the lectures, they were given a problem set about the symmetric group, and filmed while they worked together for 90 minutes on the problems. The footage captured their hand gestures, body language, and boardwork, in addition to their conversations. One week later I conducted one-on-one follow-up interviews with each, asking about the problems and their experience of understanding or not understanding those problems.

The comparison of the hand gestures and boardwork of my original lectures and the students' work will hopefully reveal insight into the type of understanding and ways of understanding of the students. This analysis is inspired by work on embodied mathematical cognition [LN00, Núñ06], and has been aided by conversations with Lawrence University professor of Education Bob Williams, and his work in this area [Wil12].

2.3. **Math-art collaborations.** Works of math-art, such as those exhibited at the Joint Meetings, often struggle to engage both contemporary mathematics and contemporary art. Often the work of a mathematician with little art training, or an artist with little math training, they can resort to craft and recreational math, and miss their mark. Yet, mathematics and art are ancient institutions of creativity and transcendence that should not remain indifferent to one another. I believe the solution is collaboration: expert mathematicians working with expert artists to create work that is relevant to both communities.

Since 2007 I have been collaborating with artists – trained fine artists, musicians, visual artists, and dancers – to create relevant math-art work that engages both contemporary mathematics and contemporary art. Work has been exhibited in galleries or performed in Portland, Baltimore, Seattle, Berlin, Sweden, Denmark, The Netherlands, and most recently India. These projects are listed on my CV right after my publications, because I believe they hold water.

For descriptions with images, video, and supplementary documentation, please visit my website. Here I will describe some activities that have come out of these pieces, and then explain just one recent project and one future project in more detail.

I have given many talks on my solo projects and collaborations, including a presentation during a 5-day workshop on math and art at the Banff International Research Station in 2011. Bringing together mathematicians and artists to speak, I've helped organize several *salons*, one in collaboration with the Math Department at the University of Copenhagen.

Elizabeth McTernan and I have written two published papers about our work [MW12, MW13], and a manifesto *"A capacity for the sublime": math and art as experience*, which is available on my website. We recently submitted a paper [MW15] on one of our pieces to the interdisciplinary journal Leonardo.

For two years I have served on the conference program committee of the annual Bridges conference, which brings together hundreds of artists and mathematicians for a conference with an exhibition. I also help referee their proceedings. In 2013 I was asked to write a report [Wol13b] on the Bridges art exhibition, for the Journal of Mathematics and the Arts. This summer, The Mathematical Intelligencer asked me to write a book review [Wol15b] of Robert Tubb's *Mathematics in 20th-century literature and art*.

Describing one piece may help to illustrate my approach. As described in Section 2.2, in 2013 I gathered written reflections from research mathematicians about their lived experience of working with group theory. In fall 2014, the collated survey responses were handed over to the Lawrence Creative Writing Club. The student writers produced six poems based on the mathematicians' comments. These were then delivered to the Lawrence improvisational music ensemble, IGLU. The group used the poems as inspiration for eight structured musical improvisations, which were presented in a concert in February 2015. A recording is available on YouTube[5]. I also wrote a report [Wol15c], to appear in The Mathematical Intelligencer, analyzing the project as a metaphor for certain aspects of mathematical practice, and as a template for future collaborations.

My next project also involves undergraduates. At the 2014 MathFest in Portland, I organized and co-facilitated a four-day collaborative evening workshop, bringing together mathematicians and local dancers. The math we learned was topological data analysis; the dance we did was structured improv; the piece was called *Bodies of Data*. During the two-week December 2015 term here at Lawrence, dance teacher Monica Rodero and I are doing a repeat of this workshop, expanded and adapted to fit the new context.

2.4. **Exposition and outreach.** Perhaps informed by my diverse travel and work experience (see the Miscellaneous section of my CV), I have a strong interest in reaching out of academia to make mathematics and the mathematical experience accessible and relevant to the world.

The first manifestation of this passion was a whimsically-titled 142-page book of essays, *My name is not Susan: a love story between mathematics and non-mathematics* that I self-published in 2009. The essays aim to relate math to real life, while reflecting on the mathematical process and culture. This was well-received, with Reuben Hersh, co-author of *The Mathematical Experience*, which won the National Book Award in 1983, telling me: "The intention behind *The Mathematical Experience* was to pull aside the veil around the life and work of mathematicians, to show and tell the rest of the world about us. Much of what you write does that, better than we did."

I occasionally write short pop essays about math topics and post them on my website. There are now about two dozen, and they come in pairs – a "short answer" and a "long answer" – geared towards different audiences. Reaching in the opposite direction, as a graduate student I wrote for the AMS Graduate Student Blog for two years, and my posts tried to pull the real world into the isolation of graduate school life.

Reaching between disciplines within academia, many of my projects and talks are interdisciplinary. The math-poetry-music project described in the previous section, for example, involved the creative writing club and a music ensemble. The audience at the performance was diverse and saw mathematics in a new light. I also organized a pre-concert panel where the poets and musicians could share their experiences of working with math.

The *Gardens of Infinity* project, at `gardensofinfinity.com`, was designed from the beginning to have as wide an audience as possible. Imagine a website of curated mathematical content, designed to inspire careful thought as well as deep reflection. The user enters an interactive and immersive online environment, is presented with five intriguing statements from Cantor's set theory, and actively navigates and follows where curiosity leads. Each statement, for example that $|\mathbb{Z}| < |\mathbb{R}|$, is presented with four options: Would you like a short and accessible explanation? Would you like a careful and rigorous explanation? Would you like to read about

[5]See `tinyurl.com/nlaat5v`

the story and characters behind this statement? Or would you rather like to ponder, to reflect on what it all means, philosophically and metaphorically?

Gardens of Infinity combines many of the themes that underlie the projects described in this document: interest in the big picture, exploration of unconventional expositional styles, a desire to augment rigor with humanism, interdisciplinary collaboration, and art and contemplation harnessed to illuminate the mathematical experience.

References

[Ano] Anonymous, *What is it like to understand advanced mathematics? Answer #1*, Quora. URL: http://www.quora.com/What-is-it-like-to-understand-advanced-mathematics (version: 2015-07-25).

[BIK11a] Dave Benson, Srikanth B. Iyengar, and Henning Krause, *Stratifying triangulated categories*, J. Topol. **4** (2011), no. 3, 641–666.

[BIK11b] David J. Benson, Srikanth B. Iyengar, and Henning Krause, *Stratifying modular representations of finite groups*, Ann. of Math. (2) **174** (2011), no. 3, 1643–1684.

[Bir79] Garrett Birkhoff, *Lattice theory*, 3rd ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, 1979.

[Bou79a] A. K. Bousfield, *The Boolean algebra of spectra*, Comment. Math. Helv. **54** (1979), no. 3, 368–377.

[Bou79b] ______, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281.

[BS14] Peter Bubenik and Jonathan A Scott, *Categorification of persistent homology*, Discrete & Computational Geometry **51** (2014), no. 3, 600–627.

[Bye10] William Byers, *How mathematicians think: Using ambiguity, contradiction, and paradox to create mathematics*, Princeton University Press, 2010.

[CIDSZ08] Gunnar Carlsson, Tigran Ishkhanov, Vin De Silva, and Afra Zomorodian, *On the local behavior of spaces of natural images*, International journal of computer vision **76** (2008), no. 1, 1–12.

[CCR13] Joseph Minhow Chan, Gunnar Carlsson, and Raul Rabadan, *Topology of viral evolution*, Proceedings of the National Academy of Sciences **110** (2013), no. 46, 18566–18571.

[DP02] B. A. Davey and H. A. Priestley, *Introduction to lattices and order*, 2nd ed., Cambridge University Press, New York, 2002.

[DP08] W. G. Dwyer and J. H. Palmieri, *The Bousfield lattice for truncated polynomial algebras*, Homology Homotopy Appl. **10** (2008), no. 1, 413–436.

[EMD12] Herbert Edelsbrunner and Morozov D., *Persistent homology: theory and practice*, Proceedings of the European Congress of Mathematics (2012), 31-50.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997.

[Har92] Godfrey Harold Hardy, *A mathematician's apology*, Cambridge University Press, 1992.

[HP99] Mark Hovey and John H. Palmieri, *The structure of the Bousfield lattice*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 175–196.

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610.

[IK13] Srikanth B. Iyengar and Henning Krause, *The Bousfield lattice of a triangulated category and stratification*, Math. Z. **273** (2013), no. 3-4, 1215–1241.

[Kel94] Bernhard Keller, *A remark on the generalized smashing conjecture*, Manuscripta Math. **84** (1994), no. 2, 193–198.

[LN00] George Lakoff and Rafael E Núñez, *Where mathematics comes from: How the embodied mind brings mathematics into being*, Basic books, 2000.

[MRS01] Mark Mahowald, Douglas Ravenel, and Paul Shick, *The triple loop space approach to the telescope conjecture*, Homotopy methods in algebraic topology (Boulder, CO, 1999), Contemp. Math., vol. 271, Amer. Math. Soc., Providence, RI, 2001, pp. 217–284.

[MMW+15] Cary Malkiewich, Mona Merling, David White, Luke Wolcott, and Carolyn Yarnall, *The user's guide project: giving experiential context to research papers*, Journal of Humanistic Mathematics **5** (2015), no. 2, 186–188.

[MW12] Elizabeth McTernan and Luke Wolcott, *Imagining negative dimensional space*, Proceedings of Bridges 2012: Mathematics, Music, Art, Architecture, Culture (2012), 637–642.

[MW13] ______, *Exquisite failure: The telescope as lived object*, Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture (2013).

[MW15] ______, *A tree calls* (2015). submitted to Leonardo.

[Mil92] Haynes Miller, *Finite localizations*, Bol. Soc. Mat. Mexicana (2) **37** (1992), no. 1-2, 383–389. Papers in honor of José Adem (Spanish).

[NHH+15] Takenobu Nakamura, Yasuaki Hiraoka, Akihiko Hirata, Emerson G Escolar, and Yasumasa Nishiura, *Persistent Homology and Many-Body Atomic Structure for Medium-Range Order in the Glass*, Nanotechnology **26** (2015).

[Nee92] Amnon Neeman, *The chromatic tower for $D(R)$*, Topology **31** (1992), no. 3, 519–532. With an appendix by Marcel Bökstedt.

[Nee00] ———, *Oddball Bousfield classes*, Topology **39** (2000), no. 5, 931–935.

[NLC11] Monica Nicolau, Arnold J Levine, and Gunnar Carlsson, *Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival*, Proceedings of the National Academy of Sciences **108** (2011), no. 17, 7265–7270.

[Núñ06] Rafael Núñez, *Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics*, 18 unconventional essays on the nature of mathematics (2006), 160–181.

[NOMPUT+15] Nina Otter, Mason Porter, Ulrike Tillmann, Peter Grindrod, and Heather Harrington, *A roadmap for the computation of persistent homology* (2015), available at `arXiv:1506.08903v3[math.AT]`.

[Pet99] Claire Petitmengin, *The intuitive experience*, Journal of Consciousness Studies **6** (1999), no. 2-3, 43–77.

[PBN06] Claire Petitmengin, Michel Baulac, and Vincent Navarro, *Seizure anticipation: are neurophenomenological approaches able to detect preictal symptoms?*, Epilepsy & Behavior **9** (2006), no. 2, 298–306.

[PRCCT13] Claire Petitmengin, Anne Remillieux, Béatrice Cahour, and Shirley Carter-Thomas, *A gap in Nisbett and Wilson's findings? A first-person access to our cognitive processes*, Consciousness and Cognition **22** (2013), no. 2, 654–669.

[Po09] Claire Petitmengin and others, *The validity of first-person descriptions as authenticity and coherence*, Journal of Consciousness Studies **16** (2009), no. 10-12, 252–284.

[Rav84] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), no. 2, 351–414.

[RJS98] Bethany Rittle-Johnson and Robert S Siegler, *The relation between conceptual and procedural knowledge in learning mathematics: A review.* (1998).

[RJSA01] Bethany Rittle-Johnson, Robert S Siegler, and Martha Wagner Alibali, *Developing conceptual understanding and procedural skill in mathematics: An iterative process.*, Journal of educational psychology **93** (2001), no. 2, 346.

[Ste14] Greg Stevenson, *Derived categories of absolutely flat rings*, Homology Homotopy Appl. **16** (2014), no. 2, 45–64.

[TT] Terence Tao, *There's more to mathematics than rigour and proofs*, What's New. URL: https://terrytao. wordpress.com/career-advice/there's-more-to-mathematics-than-rigour-and-proofs/ (retrieved: 2015-07-25).

[TKH+15] Dane Taylor, Florian Klimm, Heather A Harrington, Miroslav Kramár, Konstantin Mischaikow, Mason A Porter, and Peter J Mucha, *Topological data analysis of contagion maps for examining spreading processes on networks*, Nature communications **6** (2015).

[WT10] William Thurston, *Thinking and Explaining*, MathOverflow, 2010. URL: http://mathoverflow.net/q/38639 (version: 2010-09-14).

[CTLZTH14] Chad Topaz, Lori Ziegelmeier, and Tom Halverson, *Topological data analysis of biological aggregation models* (2014), available at `arXiv:1412.6430v3[q-bio.QM]`.

[AVQ15] Alexandra Van-Quynh, *A phenomenological approach to the intuitive aspect of mathematical practice* (2015). to appear in Teorema.

[PV94] Pierre Vermersch, *L'entretien d'explicitation*, ESF, 1994/2003.

[Wil12] Robert F Williams, *Image schemas in clock-reading: Latent errors and emerging expertise*, Journal of the Learning Sciences **21** (2012), no. 2, 216–246.

[Wol13a] F. Luke Wolcott, *On contemplation in mathematics*, Journal of Humanistic Mathematics **3** (2013), no. 1, 74–95.

[Wol13b] ———, *The Bridges Enschede 2013 Exhibition of Mathematical Art*, Journal of Mathematics and the Arts **7** (2013), no. 3-4, 150–157.

[Wol14] ———, *Bousfield lattices of non-Noetherian rings: some quotients and products*, Homology Homotopy Appl. **16** (2014), no. 2, 205–229.

[Wol15a] ———, *Variations of the telescope conjecture and Bousfield lattices for localized categories of spectra*, Pacific J. Math. **276** (2015), no. 2, 483–509.

[Wol15b] ———, *Mathematics in 20th-century literature and art, by Robert Tubbs* (2015). to appear in The Mathematical Intelligencer.

[Wol15c] ———, *Math, poetry, and music improvisations: an interdisciplinary collaboration* (2015). to appear in The Mathematical Intelligencer.