What are Cohomological Bousfield Classes?

a long answer

To a topologist, a space is anything that has a notion of “closeness” and a well-defined notion of “dimension”. We don’t care about stretching, squeezing, or pinching, but *do* care about tearing or cutting. For example, a one-dimensional space is something you can draw with a pencil on a piece of paper. To a topologist, the numerals $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}$, and $\mathbf{7}$ are all the same; the numerals $\mathbf{4}, \mathbf{6}$, and $\mathbf{9}$ are all the same; and the numeral $\mathbf{8}$ is unique. If you twist a $\mathbf{1}$ into a $\mathbf{3}$, it’s still considered the same space, but if you cut the $\mathbf{8}$ and make it a $\mathbf{6}$, you’ve changed the space.

Using this lens, some topologists study the properties of specific spaces, and others study the collection of all spaces as a whole. I do the latter. Consider all the different possible spaces, of all possible dimensions, together - all the one-dimensional ones, the two-dimensional ones, . . . , the 15-dimensional ones, etc. - call this $\mathcal{C}$. What does this collection $\mathcal{C}$, as a whole, look like? What sort of properties does it have?

Actually, we take one more step of abstraction, and consider a generalization of the notion of “space” to allow for negative dimensions. The objects are called “spectra” - a spectrum sort of behaves like a space, but now can have a negative dimension. So we consider the collection of all spectra, call it $\mathcal{S}$, rather than the collection $\mathcal{C}$. And we try to understand the global properties and structure of this collection $\mathcal{S}$. (The motivation for considering $\mathcal{S}$ instead of $\mathcal{C}$ is partly aesthetic - the math is more elegant - and partly practical, since certain tools become available only when negative dimensions are allowed.)

These questions are hard, but mathematicians have a technique for making them somewhat easier. We zoom out. We zoom out, and the picture gets a little fuzzy. Things - in this case spectra - that are not the same, start to look the same. Some of them clump together and are identified as equivalent. So instead of considering the collection of all spectra, we consider the collection of “Bousfield classes” of spectra - the different fuzzy clumps of spectra.

When we do this, the picture becomes much nicer, and easier to understand. In fact, we see a “complete lattice”, and within that various nice structures, for example a “distributive frame”, and some “Boolean algebras”.

Well, there are actually two ways of zooming out. One way yields “homological Bousfield classes” (HBCs) - the collection of HBCs make up the complete lattice just mentioned. The other way yields “cohomological Bousfield classes” (CBCs). The collection of CBCs doesn’t quite make a lattice, and for various reasons hasn’t been studied as much.

I’m undertaking to investigate the collection of CBCs.

It is often the case, in math, that when we start with some particularly nice object and generalize or abstract, the resulting object loses some of the simplicity and niceness of the original. The situation here is an example of this. It turns out that every HBC is in fact a CBC - so the CBCs make up a larger, more general, and less-well-behaved collection. And yet, they do preserve many nice properties of the HBCs. So I’m teetering on the edge of chaos, trying to use what’s known about HBCs and spectra, to understand these strange and, so far, unpredictable CBCs.

What sort of techniques am I using to understand the CBCs? As I mentioned, every HBC can be thought of as a CBC, so there's a way of embedding the collection of HBCs into the collection of CBCs. There are several other functions, some of which I've come up with, that input an HBC and output a CBC, or vice versa. These various functions preserve various properties that HBCs or CBCs can have, so they provide a way of moving between the two different collections, mapping out the objects and their properties.

For these functions to be useful, and for a deeper understanding of the kinds of things to expect when looking at the global structure of the collection of all CBCs, it's necessary to do some specific CBC computations. So recently I've been trying to compute the CBCs of various specific well-behaved or well-understood spectra.

The hope is that, by combining a deep knowledge of various specific CBCs with broad structural tools like these functions between the HBCs and CBCs, I'll be able to reveal the secrets hidden within the collection of CBCs.

Luke Wolcott
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www.forthelukeofmath.com