What are Cohomological Bousfield Classes?

a short answer

To a topologist, a space is anything that has a notion of "closeness" and a well-defined notion of "dimension". We don't care about stretching, squeezing, or pinching, but *do* care about tearing or cutting. For example, a one-dimensional space is something you can draw with a pencil on a piece of paper. To a topologist, the numerals $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}$, and **7** are all the same; the numerals $\mathbf{4}, \mathbf{6}$, and **9** are all the same; and the numeral **8** is unique. If you twist a **1** into a **3**, it's still considered the same space, but if you cut the **8** and make it a **6**, you've changed the space.

Using this lens, some topologists study the properties of specific spaces, and others study the collection of all spaces as a whole. I do the latter. Consider all the different possible spaces, of all possible dimensions, together - all the one-dimensional ones, the two-dimensional ones, . . . , the 15-dimensional ones, etc. - call this $\mathcal{C}$. What does this collection $\mathcal{C}$, as a whole, look like? What sort of properties does it have?

These questions are hard, but mathematicians have a technique for making them somewhat easier. We zoom out. We zoom out, and the picture gets a little fuzzy. Things - in this case spaces - that are not the same, start to look the same. Some of them clump together and are identified as equivalent. So instead of considering the collection of all spaces, we consider the collection of "Bousfield classes" of spaces - the different fuzzy clumps of spaces.

When we do this, the picture becomes much nicer, and easier to understand. In fact, we see a "complete lattice", and within that various nice structures, for example a "distributive frame", and some "Boolean algebras".

Well, there are actually two ways of zooming out. One way yields "homological Bousfield classes" (HBCs); the collection of HBCs make up the complete lattice just mentioned. The other way yields "cohomological Bousfield classes" (CBCs). The collection of CBCs doesn't quite make a lattice, and for various reasons hasn't been studied as much.

I'm investigating the collection of CBCs.

It is often the case, in math, that when we start with some particularly nice object and generalize or abstract, the resulting object loses some of the simplicity and niceness of the original. The situation here is an example of this. It turns out that every HBC is in fact a CBC - so the CBCs make up a larger, more general, and less-well-behaved collection. And yet, they do preserve many nice properties of the HBCs. So I'm teetering on the edge of chaos, trying to use what's known about HBCs and spaces to understand these strange and, so far, unpredictable CBCs.

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