What are Cohomological Bousfield Classes?

a technical answer

The context in which I'm working is not actually spaces or spectra, but rather that of an axiomatic stable homotopy category [HPS97], more specifically a monogenic one. The category of spectra is an example of a monogenic stable homotopy category, but there are others. The other main example I'm interested in is the derived category of a ring $D(R)$. An axiomatic stable homotopy category is, loosely, a triangulated category that has a symmetric monoidal product $\wedge$, arbitrary coproducts, and a distinguished element S^0 that is a small, weak generator.

In a stable homotopy category $\mathcal{C}$, a homology functor is an additive covariant functor from $\mathcal{C}$ to abelian groups that commutes with arbitrary coproducts and is exact (takes triangles to long exact sequences). A cohomology functor is contravariant, exact, and takes coproducts to products. Given an object X in $\mathcal{C}$, X defines a homology functor $X_*(-) = [S^0, X \wedge -]_*$ and a cohomology functor $X^*(-) = [-, X]_*$.

The homological Bousfield class (HBC) of X is the collection of X_*-acyclics, i.e.

$$\langle X \rangle = \{Y \ : \ X_*(Y) = 0\}.$$

The cohomological Bousfield class (CBC) of X is the collection of X^*-acyclics, i.e.

$$\langle X^* \rangle = \{Y \ : \ X^*(Y) = 0\}.$$

We say X and Z are (homologically) Bousfield equivalent if $\langle X \rangle = \langle Z \rangle$. The collection of equivalence classes under this equivalence relation is the (homological) Bousfield lattice. It is in fact a complete lattice, partially ordered by reverse inclusion. This equivalence relation has proved very useful for understanding the category of spectra, and the derived category of a Noetherian ring.

If we instead consider the equivalence relation given by equating X and Z when $\langle X^* \rangle = \langle Z^* \rangle$, much less is known. The collection of cohomological Bousfield classes has a partial order and a minimal element, but in general no maximum. It has a join, but in general no meet.

It turns out that, in a monogenic stable homotopy category with a good notion of Brown-Comenetz duality (including Spectra and $D(R)$ when R is connected graded), every HBC is a CBC. Specifically, $\langle X \rangle = \langle IX^* \rangle$. So the CBCs are a generalization of HBCs. Perhaps. One question I'm investigating is whether every CBC is also an HBC. If the answer is yes, then it may yield an interesting tool for relating homology and cohomology functors. If the answer in no, then the CBCs will give previously unknown examples of thick and localizing subcategories, which would also be useful and interesting.

In the case of a Noetherian ring R, I've shown that every object X in $D(R)$ has $\langle X \rangle = \langle X^* \rangle$. There are counterexamples showing this is not always the case. For example, in the category of spectra, $\langle S^0 \rangle \neq \langle (S^0)^* \rangle$ (since the cohomotopy groups are not all trivial).

What sort of techniques am I using to understand the CBCs? There are several maps I've come up with for moving back and forth between the collection of HBCs and the collection of CBCs. (For example, I've found a pair of adjoint functors that do this.) These various maps preserve various properties that HBCs or CBCs can have, so they provide a way of moving between the two different collections, mapping out the objects and their properties.

For these functions to be useful, and for a deeper understanding of the kinds of things to expect when looking at the global structure of the collection of all CBCs, it's necessary to do some specific CBC computations. So recently I've been trying to compute the CBCs of specific objects. Some computations have been done by Hovey [Hov95] in the category of spectra. Recently I've been working in $D(\Lambda)$, where Λ is a truncated polynomial algebra on infinitely many generators, as in [DP08]. The ring Λ is an interesting non-Noetherian ring, and $D(\Lambda)$ bears some similarity to the category of spectra.

The hope is that, by combining a deep knowledge of various specific CBCs with broad structural tools like these maps between the HBCs and CBCs, I'll be able to reveal the secrets hidden within the collection of CBCs.

References

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Luke Wolcott
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www.forthelukeofmath.com