What is the Derived Category of a non-Noetherian Ring?

a long answer

A ring is a mathematical object that has a good notion of addition and multiplication. For example, the integers form a ring, and the real numbers form another ring. The hours on a clock also form a ring, where $8+6=2$, because six hours after eight 'o' clock is two 'o' clock. Similarly, we could multiply $3\times 5=3$, since on the clock $15=3$. The ring is called "commutative" if the multiplication is commutative.

Given a commutative ring R, there's a way to construct a very elaborate math world, or "category," called the derived category of R and denoted $D(R)$. The objects in this category are quite complex, but by increasing complexity new phenomena and tools emerge. For example, in a derived category there are things called "exact triangles," which consist of three objects with functions between them forming a triangle. These are very useful for computations. There is also a way of multiplying objects, called the "smash product," a way of gluing together objects, and a way of simultaneously looking at all the functions between any two objects. The category $D(R)$ also lends itself to well-behaved interactions with other, much different categories. The derived category of a ring is considered by many mathematicians to be a very beautiful and elegant world in which to work.

It's sort of like baking a pie. You might start with cherries, which are tasty in a straightforward way. You go through an elaborate process to construct the pie, which is derived from the cherries, but is really something quite special on its own. Pie is very rich. As you eat it, you notice all its different colors, textures, flavors, and smells. Pie interacts nicely with ice cream.

If you started with apples instead of cherries, you'd end up with a different pie. This is like changing the ring R. Besides studying the derived category of a specific ring, you can consider a collection of different rings, and look at the corresponding derived categories. Pies can be made out of lots of things, and most fruit pies have common features.

If a ring is "Noetherian," it satisfies a certain technical condition, that says, in a certain sense, that the ring is not too big. (The term pays homage to Emmy Noether, described by Einstein as the most important woman in the history of mathematics.) A ring is "non-Noetherian" if it is not Noetherian.

I'll give an example that won't really make sense, but might visually demonstrate the difference. The rings

$$\mathbb{Z}[x_1],\ \ \mathbb{Z}[x_1,x_2],\ \ \text{and}\ \ \mathbb{Z}[x_1,x_2,x_3]$$

are all examples of Noetherian rings. On the other hand,

$$\mathbb{Z}[x_1,x_2,x_3,\dots]$$

is a non-Noetherian ring, where the "$\dots$" indicate that the list continues forever.

All the derived categories of the Noetherian rings have common properties and behaviors. They're not the same, but as far as I'm concerned they're more the same than they are different. The derived category of a non-Noetherian ring behaves quite differently. Because it is not bounded in the way Noetherian rings are, the ring is more complex, and the resulting derived category is much harder to understand. This is one of my current research directions.

My approach is to look at the “Bousfield classes” of objects in these different categories. (See the essay “What are Cohomological Bousfield Classes?” for more information.) In essence, while looking at a derived category we zoom out and make the picture fuzzy. Certain objects that are not the same start to look the same, and the picture becomes somewhat simpler.

If we start with a Noetherian ring, construct its derived category, and then zoom out, the picture we get is quite nice. We see the “Bousfield lattice,” which is in fact a “Boolean algebra” - a symmetrical sort of crystal with lots of structure to it.

Just like the integers have their prime numbers, every ring has a “prime spectrum.” If you consider the prime spectrum as a set, called $\mathsf{Spec\ R}$, (like considering the set of prime numbers,) you can start to pick out subsets. Some subsets will be contained in others, and, based on containment, the collection of all subsets of $\mathsf{Spec\ R}$will form a lattice. It turns out that this lattice, of subsets of $\mathsf{Spec\ R}$, is *exactly the same* as the Bousfield lattice we see when we zoom out from the derived category of a Noetherian ring. So the derived category contains enough information to basically reconstruct the original ring. This might be like being able to extract a seed from the baked pie, and use that seed to regrow the original fruit tree.

But this only works if the original ring is Noetherian. If the ring is non-Noetherian, when we construct the derived category and zoom out, the picture gets simpler but not quite as nice. There is not necessarily a lattice. It doesn’t necessarily correspond to the prime spectrum of the ring.

Why not just give up, and post a sign warning ”Here be dragons”? That’s not how mathematicians work. Once we understand something, we push farther and ask harder questions. From the perspective of Bousfield classes, we know everything there is to know about the derived category of a Noetherian ring. It’s quite straightforward, and almost *too* simple. The non-Noetherian case offers a chance at new complexities and new connections.

In particular, there’s an important topological category (the “stable homotopy category”), not quite the derived category of a ring but profoundly similar to one, that falls into the non-Noetherian case. So an investigation of the derived category of a non-Noetherian ring bridges the two fields of algebra and topology.

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