What is the Derived Category of a non-Noetherian Ring?

a technical answer

The derived category of a commutative ring R is a classical construction in homological algebra. The objects are (unbounded) chain complexes of R-modules. The morphisms are equivalence classes of chain maps, where weak equivalences (those that induce isomorphisms of homology groups) are inverted. This category is triangulated: cofibers are given by the cone construction. The total tensor product and $\mathbb{R}Hom$ construction give $D(R)$ a symmetric monoidal structure that is compatible with the triangulation. In fact, $D(R)$ is a monogenic stable homotopy category [HPS97]. Coproducts are taken degree-wise, and the sphere object S^0 is a single copy of R in degree zero.

The Bousfield class of an object X in $D(R)$ is

$$\langle X \rangle = \{Y \;:\; X \wedge Y = 0\}.$$

We say X and Z are Bousfield equivalent if $\langle X \rangle = \langle Z \rangle$. This gives an equivalence relation, and we study the equivalence classes.

When R is Noetherian, Neeman [Nee92] showed that the Bousfield classes form a lattice that is in one-to-one correspondence with subsets of the prime spectrum $\mathsf{Spec}\,\mathsf{R}$ of the ring R. Furthermore, every localizing subcategory (a triangulated subcategory closed under retracts and coproducts) is a Bousfield class, so this gives a classification of all localizing subcategories. In addition, the thick subcategories (triangulated subcategories closed under retracts) of finite objects (those in the thick subcategory generated by the sphere object) are in one-to-one correspondence with those subsets of $\mathsf{Spec}\,\mathsf{R}$ that are closed under specialization. In fact, [HPS97] extend these results to a larger class of axiomatic stable homotopy categories for which the ring $\pi_*(S^0) = [S^0, S^0]_*$ is Noetherian.

For a non-Noetherian R, the Bousfield classes do not necessarily form a lattice. In fact, it is not known whether or not there is even a set of Bousfield classes. (In the case where R is countable, we do know there is only a set.) The main non-Noetherian ring I am studying is

$$\Lambda = \frac{k[x_1, x_2, \dots]}{(x_1^{n_1}, x_2^{n_2}, \dots)},$$

where k is a countable field, all $n_i \geq 2$ are fixed, and the x_i are graded so that Λ is graded-connected and graded-commutative (for example, take $|x_i| = 2^i$).

Because Λ is countable, there is a set of Bousfield classes, and these form a lattice. The derived category $D(\Lambda)$ has been studied in [DP08]. There it is shown that the Bousfield lattice of $D(\Lambda)$ has cardinality $2^{2^{\aleph_0}}$. Note that the prime spectrum of Λ has only two elements. This is one example of how the non-Noetherian condition results in more complex behavior.

The topological stable homotopy category, consisting of spectra, is also a monogenic stable homotopy category. It has a non-Noetherian ring $\pi_*(S^0)$ that, like Λ, is locally finite, graded-connected, and graded-commutative, with few prime ideals. So investigating $D(\Lambda)$, and similar categories, has the potential to inform our study of the category of spectra.

References

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