Intuition in mathematics: a perceptive experience

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Abstract

This study applied a method of assisted introspection to investigate the phenomenology of mathematical intuition arousal. The aim was to propose an essential structure for the intuitive experience of mathematics. To achieve an intersubjective comparison of different experiences, several contemporary mathematicians were interviewed in accordance with the elicitation interview method in order to collect pinpoint experiential descriptions. Data collection and analysis was then performed using steps similar to those outlined in the descriptive phenomenological method that led to a generic structure that accounts for the intuition surge in the experience of mathematics, which was found to have four irreducible structural moments. The interdependence of these moments shows that a *perceptualist* view of intuition in mathematics, as defended by Chudnoff (Chudnoff, 2014), is relevant to the characterization of mathematical intuition. The philosophical consequences of this generic structure and its essential features are discussed in accordance with Husserl's philosophy of ideal objects and theory of intuition.

Keywords: Phenomenology, intuition, mathematics, introspection, interview of elicitation, phenomenological reduction.

A- Introduction

The essential role of intuition in the foundations of mathematics has been debated since the ancient Greek period. Though the purpose of this article is not to review the work of authors who wrote about this issue, it is nevertheless necessary to note that for many modern mathematicians (Poincaré, 1905; Poincaré, 1908; Hadamard, 1945; Frege, 1971; Pólya, 1954, Pólya, 1980; Gödel, 1985; (Rota, 1997; Hersh, 2013; Gowers, 2002), intuition is a founding, inescapable and real mode of knowing. For all these mathematicians it remains though that any hypothesis or statement "obtained" through intuition is to be accompanied later on by a proof. To borrow Poincaré's words, in mathematics intuition is the instrument of invention while logic is that of proof (Poincaré, 1905).

A central attribute of intuition is its immediacy. It occurs (or seems to occur) beyond any discursive reasoning and is not reducible to a mental computational process. This feature makes an accurate description of intuition difficult to achieve due to the time discontinuity between the *before* and the *after* of the insight upsurge (Petitmengin, 2001; Chudnoff, 2014). The immediacy of the insight is often acknowledged as being accompanied by feelings of necessity and evidence[1]: intuition would be this faculty of human understanding for perceiving essences and reaching truth (Poincaré, 1905; Hersh, 2013; Author, 2015a). In their pioneering works on the philosophy of mathematical practice and mathematician's psychology, Poincaré

[1] Frege uses the term of self-evidence to describe the axioms given in the intuition (Frege, 1971)

(1905) and Hadamard (1945) propose general features for the description of the moment of insight in mathematics and give tentative accounts of its upsurge. In his famous four-step model of mathematical activity, Poincaré refers to a subliminal work of consciousness in which mathematical intuition plays a decisive role. Hadamard deepens Poincaré's description and assumes an *abductive* character of mathematical thinking, wherein the mathematician's mind has a subconscious activity that allows several combinations of mathematical relevance, amongst which the best is chosen. These works represent decisive steps in the development of the phenomenology of mathematical practice. However, they are limited when it comes to a "full" description of insight arousal and genesis. Indeed, Poincaré, who used the metaphor of Epicurus's hooked atoms to explain his model of intuition in mathematics, confessed that *"[his] comparison* [*i.e.* the hooked atom metaphor] *was very crude but that [he] cannot see how [he] could explain his thought in any other way*" (1905).

Kurt Gödel strongly defended the existence of mathematical intuition to which he gave a central place in the foundations of mathematics. Gödel draws parallels between mathematical intuition and sense perception: through intuition humans have the ability to transcendently perceive axioms and mathematical truths in a manner analogous to their perception of physical objects. Gödel also sought a way to describe the experience of mathematical insight. He found in Husserl's phenomenology a promising method as, according to Husserl, in adopting a certain mental posture, a subject enters into a new state of consciousness that allows her to re-orient her awareness towards a present moment of experience and describe it in detail. It is "*a procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us"* (Gödel, 1985, p. 383).

Following the path opened by these famous mathematicians, we intend to:

(a) propose a phenomenological description of the intuitive aspect of mathematical practice;

(b) investigate the moment of insight in mathematics by applying a psychological research method to genuinely lived experiences of mathematics with the final aim of accounting for an essential structure of the emergence of intuition in mathematics (given that it has one) and then to study the philosophical implications of this structure.

The first objective was pursued by collecting the responses of contemporary mathematicians to a survey concerning their private practice of mathematics (Author, 2015a; Author, 2015b). To briefly summarize the study, we gathered mathematicians' answers to a questionnaire that inquired about their methodology, the manner in which they approach and solve problems, and the place they consider to be occupied by intuition[2] in the progress of their mathematical knowledge. We then categorized these first-person verbal reports and searched for invariants in mathematicians' answers. This enabled us to put forward an intersubjective structure of the intuitive aspects of mathematical research. In addition, by aligning our results with the philosophical works of Hadamard and Poincaré (Hadamard, 1945; Poincaré, 1908; Poincaré, 1920), as well as recent research on the phenomenology of mathematics

[2] 'Intuition' is understood in the sense of 'illumination' or 'moment of insight'. We will further discuss this notion later in the text.

(Corfield, 2004, Govers, 2002; Hersh, 1999; Hersh, 2013; Rota, 1997), we identified an inherent scheme of mathematical discoveries. The survey outcome was a powerful means to, first, evidence a common methodology in the manner mathematical problems are approached and ultimately solved, and second, to identify several facets of intuition in mathematical practice common to contemporary mathematicians. But this phenomenological account remained_incomplete. Indeed, the description of the unfolding of the moment of insight in mathematics was not assessed. To remedy this, it was necessary to unveil the details of private mental and bodily processes and gestures occurring in singular moments of insight for different mathematicians, with the aim of extracting from them a generic structure of mathematical intuition genesis. This project was initiated with due reference to Poincaré's work and Gödel's project and encouraged by Rodin's historical argumentation on the role played by intuition in today's mathematical concepts' development (Rodin, 2010).

The fundamental difference between objectives (a) and (b) lies in the fact that to fulfill (b), we had to move from a mathematician's general representations of intuition in mathematics - which were scrutinized and reported in (Author, 2015a) - to a focus on first-person descriptions of *singular* intuitive experiences *effectively lived* by mathematicians. In a practical manner, instead of letting mathematicians freely report on what "*they think they generally experience/do when they have intuition*", we utilized guided interviews that allowed mathematicians to re-enact a specific past experience of theirs and to verbalize the acts that brought it about.

A recent phenomenological study of intuitive experiences in different fields of knowledge used a specific protocol inspired by Husserl's phenomenology – the elicitation interview (EI) (Petitmengin, 2001; Petitmengin, 2006). Thanks to this, in several fields of knowledge[3], Petitmengin could approach an examination of the intuitive process by following the sequence of gestures associated with the emergence of the intuition.

Our investigations were discovery-oriented for we did not believe a generic structure inevitably existed for the practice of mathematics. We employed an inductive methodology by starting with descriptions of experiences that were representative of the context of discovery in mathematics, with the aim of acquiring an understanding of the intuition arousal phenomenon. In keeping with fundamental phenomenological procedure, our experimental frame was free from philosophical presuppositions or theories about mathematical intuition and our intent was not to address the question of the ontology of mathematical objects.

The investigation reported here has employed, similar to Petitmengin, the elicitation interview (EI) to collect subjective descriptions of intuitive experiences in mathematics. Details of the EI protocol are presented in the Methodology section. We applied the attitude of the phenomenological reduction, within Husserl's philosophical tradition, at two levels of the research process:

i) to obtain descriptive reports on what actually happened during an intuitive experience in mathematical research. We intended to ground the investigation in concrete and 'naïve' descriptions given by several contemporary mathematicians in

[3] In Petitmengin's study, the case of intuitive experiences in mathematics was not examined.

their common sense mode of understanding. We wanted to avoid discussions based on creative reconstructions of mathematical theories' developments[4];

ii) to proceed to an analysis of these descriptive reports, exempt from preconceptions about the phenomenon under study. In keeping with the attitude of the phenomenological epoché we studied the way the situations presented themselves to the mathematicians without judging their veracity from an objective (i.e. natural attitude) perspective.

In this way, our methodology came close to what is carried out in psychological investigations that use the descriptive phenomenological psychological method (DPPM) as developed by Giorgi (Giorgi, 1970; Giorgi, 1985; Broomé, 2011).

We conducted several elicitation interviews in order to assist and guide mathematicians in the introspection of their own a *genuine lived* experience. This allowed us to unfold the early stages of the cognitive processes of mathematical intuition that were not, before the interviews, accessible to the mathematicians. Section B describes our experimental methodology.

Once the interviews were recorded, the descriptions were transcribed. The texts were subsequently honed by disregarding comments on the experiences, context and common beliefs on intuition. They were then reorganized in order to recover the chronological unfolding of the different intuitive experiences. The course of action for the simplification of the initial descriptions that permitted us to obtain (what we label) *pure descriptive texts* is detailed in Section C. The analysis of the latter was conducted in accordance with a second-person perspective, inspired by the descriptive phenomenological method that allowed, from each individual's intuitive experience, the extraction of a general structure reflecting the phenomenon, as effectively experienced by the participants. Afterwards, an intersubjective comparison of the different general structures allowed an eidetic variation and led us to propose a *generic* structure for the emergence of a feeling of intuition in mathematics. This structure is composed of four essential moments. We detailed this analysis process in Section C.

In Section D we discuss and compare our results with past and recent investigations on intuition in mathematics. Specifically, we show how a *perceptualist* view of mathematical intuition is conducive to a phenomenological analysis and discuss the philosophical implications of this view. Finally, we examine the constituting role of the intuitive experience in mathematics and its resonance with Husserl's theory of intuition.

B- Methodology

1- The elicitation interview (EI)

Our methodological challenge was to disclose information and details on singular lived experiences that the subjects themselves did not know and/or were not conscious

[4] Quine defends the investigation of the context of discovery in sciences for a reliable development of an epistemology of scientific knowledge and employs the inspired expression of *make-believe* (Quine, 1969). See also (Author, 2015a) for a discussion on the philosophical relevance of studying the context of discovery in mathematics.

of. As stated in the introduction, EI was our means to achieve goal (b). The subjects retrospectively accessed the content of their lived experiences in an act called *evocation*. Evocation brings the subject in contact with her past lived experience and puts her in a position where she can speak, in an embodied way, of "being present to herself" as the experience is evoked. The recollection relies on a re-enactment of the lived experience: in the evocation posture the passive memory is awakened; this allows access to details of the lived experience that were unconsciously retained by the subject.

Vermersch developed the EI protocol based on Husserl's conception of consciousness (Vermersch, 1994; Vermersch, 1999). EI relies on both the fact that a subject permanently retains memories of lived experience and can solicit these memories[5]. It is nowadays often used to collect detailed descriptions of the pre-reflective content of lived experiences of a given sort and to assess the unnoticed microdynamical structure of those experiences (see for instance Petitmengin *et al*., 2013; Bitbol & Petitmengin, 2013; Depraz *et al*., 2003; Depraz, 2014; Petitmengin & Lachaux, 2013 and, for earlier studies, Varela *et al*., 1993 and Varela, 1996). This interview protocol allows for the gathering of first and second-person data on subjective lived experiences through careful guidance. The collected data consist of fine-grained descriptions of the content of an action like listening to music, meditating, contemplating, memorizing a text or a number series, to cite just a few examples. Each EI session begins with the subject entering in close contact with her past experience, in the act of evocation. The interviewer's questioning stimulates the interviewee's passive memory. Most of the time it is the reflective component of the evoked experience that comes out first during an EI. However, as the subject is guided, they become aware of pre-reflective material, thus gaining an opportunity to verbalize it with precision[6]. The term *pre-reflective* is understood in the sense detailed by Vermersch (1994). *Reflective* consciousness is distinguished from *pre-reflective* consciousness according to the following: *reflective* consciousness is what one knows by knowing one knows it, *i.e*. what one is "already aware of". *Pre-reflective* consciousness is what one knows without knowing one knows it. Material that is pre-reflectively conscious is not something lost or absent; it is content of which one is not yet self-aware, content that is ignored until one is able to arouse it and become receptive to it. EI makes this pre-reflective consciousness, or at least part of it, accessible.

One of the strengths of the EI method is that it focuses on a *singular* lived experience; it avoids the natural tendency of "*speaking in general*" that usually refers to generic situations. For instance, we might be interested in investigating "how it was when I started brushing my teeth this morning" and not "how it is when I start cleaning out my desk". EI refers to a *particular* slice of life, rather than to a *class* of life experiences. During the act of evoking a past lived experience, the subject's attention is moved from the content of the experience to her inner processes. Thus, a conscious

[5] A full article would be necessary to properly describe the EI methodology and is not the purpose of the present article. Vermersch's EI theory involves the phenomenon of *retention* and the way that what has been unintentionally retained can be awakened if the passive memory is properly stimulated. For a detailed description of the cycle that starts with retention and ends with the recollection/remembering, see (Vermersch, 2006).

[6] We must remain humble about the expression "with precision". Indeed, during an EI the interviewee may face the frustration of not finding the proper words to describe her inner states or perceptions. We return to this point later in the text.

description in acts of the experience is made possible, revealing what we label the unnoticed micro-dynamics of the experience. To grasp the description of the experience is to unveil the temporal development of the acts, as well as its perceptive sides (such as images, sounds, colors (if relevant), and internal states). Further details on the EI Protocol are offered in sub-section 2.

An essential remark must be made about the attitude adopted by the interviewer. For the fulfilment of EI, the interviewer must practice an *epochè*: the questioning must be free from any contamination from beliefs or conceptual frameworks on the studied experience. The interviewer adopts the attitude of the phenomenological reduction, by bracketing everyday knowledge and presuppositions about the experiences of the subjects. In practice, this means that the researcher should avoid asking leading questions. The questions, "*and at that moment, were you experiencing any cold?*" and, "*and at that moment, what were you experiencing?*" naturally elicit quite different responses. If we are to learn of the interviewee's subjective experience, we cannot ask questions founded in our own assumptions regarding the experience. Rather, we must only ask what the subject experiences/feels without inducing any categorization or qualification. Thus, only open and non-inductive questions are allowed.

In addition, we would like to stress the fact that we *do not* ask (or even expect) the interviewee to adopt the phenomenological reduction. As is the case with the DPPM, the research participants remain in the natural attitude, giving an account of their experience as it was lived by them within their everyday mode of understanding (Broomé, 2001; Giorgi 1985; Giorgi, 2009). It is the researcher's duty to adopt the phenomenological reduction in order to grasp the participants' (common sense) point of view.

Finally, it is remarkable that the evocation act creates a particular link between the subject and her past experience. Evocation operates on a *letting come* mode and induces for the subject a feeling of familiarity and recognition with what is being re-enacted (Petitmengin & Bitbol, 2009 and www.grex2.com).

2- EIs of mathematicians: re-enactment and description.

For the apparently unpredictable - and introspectively opaque - experience of an intuition, EI presented itself as an efficient tool for the investigation of this particular moment. It allowed us to proceed to an accurate inquiry into the mathematical insight of the unfolding of several intuitive experiences genuinely lived by contemporary mathematicians.

With the elicitation interview, the aim was to draw mathematicians' attention to the dimension of their own subjective experience – *i.e.* when they actually "do math" – by asking them to return to a past experience, re-enact that experience, and ultimately describe it. We believe we were able to assess the mathematicians' private cognitive activities and thinking processes along with their personal states and, if present, their bodily actions; that is, we think we could elucidate what occurred during their singular experience of mathematical illumination. Aside from being in a phenomenological attitude while performing the EI, the interviewer must by all means avoid asking the subjects to *explain* or *justify* what happened during her experience. We sought to determine *how* things were done, rather than *why* they were done; in other words, we were looking for raw facts as opposed to explanations or justifications. The interviews were rather long processes, taking up to one hour or so. EI in general requires rigorous and persistent efforts from both the interviewer and interviewee. On the one hand, the natural tendency of the latter is to report on the context and the *why* of the experience, as well as to flee to abstract levels and to escape from the attitude of staying in close contact with the specific experience. As a direct consequence, the interviewer is tasked with keeping the interviewee within the limits of her experience. This necessitates simultaneous firmness and gentleness in order to help the interviewee adopt an embodied speech and become aware of the pre-reflective side of the experience. The interviewer must arrive at an elucidation of the successive cognitive gestures of the experience (the successive steps) but also at a description of the sensations and perceptions that took place during this experience.

In order to properly initiate our investigation, some preliminary conditions had to be established with the participants. First, prior to the interviews, it was agreed that the term intuition would be understood as *a direct and immediate vision of a reality – immediate cognition of a truth without the use of reasoning*"[7]. This definition of intuition was chosen because it is close to what Poincaré calls *illumination* in his chronological four-step model of mathematical work[8] (Poincaré 1908). This notion of illumination is a key reference in the phenomenology of mathematics; its suddenness and unexpectedness are two features researchers in mathematics are familiar with. That may why, for the great majority of the mathematicians we interviewed, recalling an intuitive experience in their recent past was not particularly difficult.

Second, once the experience was chosen, the mathematicians were asked to write down and freely describe (meaning without any specific guidance) the main features of the experience they selected. This step turned out to be crucial for both interviewer and interviewee. It allowed the interviewer to assess the correct understanding of the term "intuition" in the choice made by each mathematician of his/her singular experience to be elicited. The written also enabled the interviewer to become familiar with each mathematician's private experience. For the mathematicians, it was an opportunity to dig into a preliminary re-enactment of their intuitive experience (even if carried out alone).

Only then would the interviews begin, with each mathematician being independently interviewed according to the EI protocol. The interviewer (the author of this article) guided mathematicians towards repeated re-enactment and evocation of their private intuitive experience. We used certain tips that help the subject to retrieve the spatio-temporal context of the experience and the sensations associated with it. For instance, we facilitated the re-enactment by provoking, through the questioning process, a *slowing down* of the stream of the experience and by systematically using the present tense in the formulation of the questions. During the interview, the interviewee's attention on the evoked experience could be stabilized by:

- inviting the mathematician to suspend any concerns other than the current interview;

- reformulating portions of the report given by the mathematician and inviting her to check on the reformulation;

[7] Dictionary of philosophy, *J. Ferrater Mora* (2001), Loyola Ed.

[8] We would have been very fortunate if we had had the opportunity to interview Poincaré after his experience in the bus, as this moment of illumination has become something of a legend in the phenomenology of mathematics.

- bringing the subject back to her description each time she stopped describing the experience and started explaining, judging it, or speaking of it in an abstract or generic manner (note: such escapes from the evocation state are easily identifiable, as they often begin with phrases such as "*in this kind of situation, I tend to…*" or "*usually when I have this kind of insight…*").

Detailed descriptions of the way the interviewee's attention is stabilized and of the kind of non-inductive questioning process conducted by the EI researcher are given in Vermersch, 1999; Petitmengin, 1999; Petitmengin, 2001; Depraz, 2014; Vermersch, 2014. Procedures like visualization techniques and breathing exercises can help the subject relax and return to the experience under evocation. This is important, as awareness of the pre-reflective content of the experience cannot be forced. Certain subjects made the comment that trying to catch the pre-reflective side of an experience was like attempting to hold a dream: the more one wants to grasp it, the further it goes away.

Note that despite the fact that achievement of a *pure* description of an experience remains an unreachable ideal, EI purports to minimize interpretative aspects often spontaneously given by the interviewee. Each time she makes comments on her experience, the interviewer re-orients the subject's attention towards the content of said experience. For example, the researcher must adopt a defiant posture regarding retroactive explanations, as these may reflect an interpretation of the experience, rather than the experience itself. The global frame of our questioning shows methodological perspectives and concerns comparable to those of the descriptive phenomenological psychological method (DPPM); in Giorgi's words: "*it is much more difficult than it may appear to describe objects of lived experiences in the way they were lived. One must avoid constructions and explanations, as well as theoretical interpretations that will be a priori explanations. The researcher must prevent the subject from generalizing and speaking in too abstract a manner. This is why the researcher's ambition is to focus the subject's attention on a specific and individuated situation in order to permit the subject to stay in the concrete and to describe it*" (freely adapted from (Giorgi, 1997)).

Before closing this sub-section, a few remarks are needed: (*a*) An EI may end when, despite repeated re-enactment of certain moments of the chosen experience, the interviewee is convinced he/she has nothing else to add to complete the description; (*b*) On the interviewer's side, it is a matter of evaluating if the level of a pinpoint description of the moment of illumination and its temporal borders has been reached. For this, while carrying out the interview, the interviewer must first circumscribe the beginning and the end of the selected experience. The so-defined duration is then to be fragmented repeatedly until the whole experience becomes fully intelligible to the interviewer; when the interviewer understands "*the story of the experience*", a satisfying *temporal fragmentation* has been obtained. In other words, the interviewer must obtain a *granular* description of the experience as a series of short successive moments (the temporal components) for which the links and connections between them is captured; (*c*) Aside from inquiring about the cognitive components of the experience, the interviewer also attempts to collect data of sensuous components. This descriptive investigation corresponds to what is called the *expansion of the qualities* of the experiential descriptions (Vermersch, 2014); (*d*) As a complement to what we noted a few paragraphs earlier on the elusiveness of historical truth, the question of the validity of introspective data is often raised. This issue has been discussed (for example, in Petitmengin & Bitbol (2009)), and the performative coherence of introspective reports collected using EI method has been empirically studied (Bitbol & Petitmengin, 2013). The results and conclusions of these studies converge towards a renewed definition of the *truth of first-person reports* where the concept of correspondence is no longer used, but rather that of performative consistency. During our interviews, we could watch a recurrent mathematicians' attitude: while describing her experience, the subject repeatedly performed adjustments in her description until she was satisfied with it. She had to reach the feeling of having produced a faithful report of the experience, exempt from the impression of having imagined or fantasized anything about it[9].

Seven mathematicians of different research domains (topology, algebra, stochastic analysis, mathematical physics, theoretical computer science, and dynamical systems) were interviewed; all are professors or researchers in universities, except one, a PhD student in the process of finishing his thesis. One to three elicitation interviews were necessary to obtain a satisfying description of the different intuitive experiences. Stimulation of the passive memory is not straightforward, especially for subjects unfamiliar with introspective practices (like meditation); consequently, the first interview often only allowed for the collection of shallow descriptions of the experience that mostly reported on the reflective content of the experience. Such a fact justified additional EI(s) on the *same* experience in order to access invisible stages of cognitive processes involved in the intuitive experience.

3- The analysis of the verbatim transcripts and its prerequisites.

Once gathered, the interviews were transcribed. For the sake of clarity and to keep the article under a reasonable number of pages, we do not give a full report of any EI transcription, as each of those texts is at least eight pages in length. Nonetheless, in order to illustrate our analysis procedure, we provide at the end of this article excerpts of the transcriptions in order to demonstrate how they were treated.

For the interviewer/researcher, it was first necessary to become familiar with the subjective content of each experience in order to get a sense of the whole. In order to apprehend such sense, the researcher adopted the attitude of the phenomenological reduction. To employ the terminology used for the DPPM methodology, which is also well suited to the attitude we adopted here, "*the researcher allows him/herself to be present to the data without positing its validity or existence. Simply being present means to look at the data as it appears in itself and in its own context without doubt or belief*" (Giorgi, 2009). Within the stance of the phenomenological reduction, the descriptions were read repeatedly until the researcher felt familiar with them. What was sought with this first step was an empathetic understanding of the experiences[10] (disregarding the mathematical aspects of it, of course).

The second step of the analysis refers to the methodology commonly used in the elicitation interview method to elucidate the micro-dynamics of experiences (Vermersch, 1999; Petitmengin, 2001; Bitbol & Petitmengin, 2013; Petitmengin & Lachaud, 2013; Depraz, 2014). It consists in a reduction of the different verbatims to

[9] It is well known that mathematicians have a very particular relationship with truth…

[10] The researcher eventually came to know by heart several passages of the transcription of each experience.

a sequence of simple propositions that express elementary relations between acts, thoughts and gestures made by the subject during her experience. The texts were thus ‘cleared out’ in order to obtain a description of the intuitive experience exempt from the context, evaluations, interpretations or justifications, which are not relevant for the effective description of the intuition unfolding (see the appendix). Even if interpretations or explanations may contain implicit meanings, these are considered as retrospective and do not contain information about the acts and gestures of the experience itself. Beliefs, opinions and theoretical knowledge about intuition were also removed from the transcriptions.

EI’s excerpts are given in the appendix. However, to briefly illustrate our simplification of the transcriptions, it may be best to give a brief example. From the piece of speech of mathematician X:

X: *I mean I guess it’s not… it’s not quite as spatial as my hand movements. That’s just… there is an “if then” statement that has just been put in the middle of, in this localization language and I… I don’t know… it like precipitates immediately, I don’t know. There is not... Yeah! It’s there and you could say “I’m trying to see how… where in the landscape it is or what that means when you bring it in this language… ” There are immediate consequences of having put that in that language. It precipitates immediately “bam!” and then you see it… I don’t know (laughing),*

we extracted the following:

- “*it’s not as spatial as my hand movements*”

- “*there is this A*”

- “*it like precipitates immediately*” and “*there are immediate consequences of having put A*”

- “*it’s there*” and “*you see it*”.

Here *A* stands for a mathematical action the mathematician X took, the details of which bear no relevance to our analysis, as we are interested in the content in acts of the experience, beyond its full mathematical details. However, during the EI the description of the mathematical context can be useful for the interviewer (to help him understand the experience (to a reasonable extent of complexity)), and for the subject (to help in re-enactment and immersion in the past experience).

The material is subsequently reorganized in order to reconstruct the chronology of each experience. During the interviews, it is common for subjects to move between various moments of the experience. Indeed, in order to deepen the description of specific moments, it is sometimes necessary to focus first on a certain time-slice around a moment t and then to “go back in time” in order to focus on the re-enactment of a time-slice around another moment, anterior to t. With this rearrangement, it is up to the researcher to assess how each singular experience is articulated in time (temporal dimension), to determine the successive steps, and to identify the diachronic[11] dimension associated to each specific step.

[11] *Diachronic* is to be understood for what is not temporal, meaning for instance the sensitive aspects of the lived experience.

The clearing-out of the raw data was performed while in the phenomenological attitude. Data were considered as they were, *i.e*. as the contents of the different experiences were given/presented to the mathematicians during their elicitation. The researcher executed repeated self-verification: while simplifying the texts and reconstructing the chronology of the experiences, the researcher prevented himself from any over-filling of the mathematician's experience. An exhaustive description of an experience is simply unreachable, as one could always (try to) descend lower and lower in granularity. Even a protocol like the EI method that aims at a pinpointed verbalization of an experience leaves gaps where descriptive elements are missing.

Thus, through this preparatory treatment of the raw data, we obtained a temporal sequential narrative made of very simple propositions that uncovered the general structure of the acts underlying each individual intuitive experience. An intersubjective comparison of the general structures was made afterwards in order to identify the *invariant properties* of the different (individual) general structures and synthesize them in a *generic* structure for insight surge in mathematics. This *generic* structure, introduced and detailed in the next section, was obtained following a procedure that possesses common traits with what Husserl calls eidetic variation, in which the objective is to extract the invariant identity of insight emergence in mathematics, apprehended in its universality *and* in its singularity (Zahavi, 2003; Depraz, 2012). Therefore, what we carried out was a kind of "sampling" eidetic variation, as we did not intend to unfold an essential insight of the intuitive experience in mathematics from a *single* description but rather after having taken all the individual general structures together in a comparative view. For this reason, in the rest of the text we will preferentially employ the term *intersubjective* (or *generic*) *structure* and more rarely that of *essential* structure.[12]

In contrast to the investigation of intuitive experiences made by Petitmengin (1999; 2001), we did not use our personal knowledge or personal experiences of intuition to assist us in the performance of the elicitation interviews or their subsequent analysis (Petitmengin, 1999). Indeed, mathematical intuition(s) are not "common experiences", in the sense that only an expert in mathematics might experience an illumination that can contribute to mathematical advances (aside from "spontaneous genius in mathematics" – but such cases are completely out of the scope of our article). So, as familiarity with mathematical intuition(s) would be unlikely among everyday people, and any attempt to guide the mathematician according to other kinds of intuitive experience (from other fields of knowledge) would have itself provoked something like a phenomenological reduction and could have complicated the results. As a corollary, the same remark holds for the analysis of the data, for which "schemes" inspired from other existing models for intuition were not used. We do not in any way insinuate that this procedure is better than Petitmengin's. We only wish to highlight the difference and offer a plausible explanation for this variation.

C- The constituents of the intuitive experience in mathematics

[12] In DPPM research, what we call here a "general structure" is called "situated structure", while what in EI terminology is called the "generic structure" is called the "general structure" by DPPM researchers.

Our analysis and the revelation of the generic structure that underlies the intuitive experiences show that the surge of intuition – or moment of insight - is a global process that is not reducible to the instant of the insight surge. The emergence of insight is characterized by a sequence of cognitive gestures that are, in some cases, accompanied by physical attitudes (meaning a certain embodiment). These are considered a full part of the intuitive experience. Our comparative analysis led to the identification of four irreducible structural moments. Together, they form the whole of the intuitive experience; each of them has an essential identity and plays an inescapable role in the experience[13].

The intersubjective structure of the moment of insight begins with a cognitive gesture of *disconnection from the outer world*. This disconnection may differ in its specificities from one experience to another, but is always present in the experiences that were elicited, thus warranting inclusion as an essential quality rather than an incidental one. This mental movement of disconnection is accompanied by the connection of the mathematician to mathematics. Hence, we call the time slice corresponding to such disconnection and subsequent link to mathematics the *preparative period*. This period is followed by a sequence of three steps. It corresponds to the unfolding of the insight itself and can be decomposed according to *i)* the moment where 'signs of portent' are perceived; *ii)* the surge of insight, and *iii)* the consequences of insight. The four following paragraphs report on the descriptions of the preparative period and the three-step sequence. Excerpts of the transcripts are cited in italics in order to illustrate the different descriptive categories of the intuitive experiences put forward by our study. The quotations are labeled according to the section they are in and numbered sequentially, to conceal mathematician's identity.

For the sake of clarity, *i.e.* to not confound the details of the intersubjective structure of the intuitive experience in mathematics and our interpretation of this structure, we tackle the latter and its phenomenological consequences in a separate section (Section D).

a- <u>The preparative phase</u>

This phase is characterized by the disconnection of the mathematician's mind from the outer world; it was identified in each individual intuitive experience.

We read: "*I'm far away from everything that's around me*" (a1) – "*I was pretty oblivious to the rest… I'm just staring at this page and I'm seeing what I have just written*" (a2) - "*I'm far away from everything and there is something that I see even if I don't have my eyes closed*" (a3).

A succeeding link of the mathematician to mathematics was then evidenced. Two different kinds of connection to math, depending on the subject, were identified:

i) The connection is voluntary: the mathematician concentrates on the problem (e.g. "*I'm staring at what is written on that page*" (a4) – "*I am concentrating myself on the terms*" (a5))

[13] In proposing these four moments/constituents of the intuitive experience, we do not assert that another researcher would necessarily identify the *same* moments but that this number of moments cannot be reduced, lest one arrive at an over-simplified or abstract generic structure (Sokolowski, 2000).

ii) The connection is involuntary: the mathematician's mind may have wandered or the subject is in a moment of mental rest but the mind gets connected to a mathematical problem (e.g. "*my thoughts had diverged a little*" (a6) or "*this is something that arrives despite my efforts to ease myself*" (a7)).

In spite of some difficulties in accessing the pre-reflective content of the experiences[14], we could identify a state common to all subjects. It could be called "a swap of worlds" (from the physical to the mathematical world), a sensation of the mind escaping from *here and now*. One mathematician described "*a sensation that thinking goes towards places where it feels very well*" (a8) immediately following the moment in which thoughts of the subject involuntary diverged from a task not related to mathematics. Another subject, oblivious to what surrounded her, accessed mathematical landscapes ("*In my mind it's like I'm visiting these little pieces of mathematical landscapes I hadn't visited before*" (a9)).

At this point, we would like to state that we do not infer from these descriptions that mathematics is necessarily an outer world that mathematicians only have access to under certain circumstances. It should be kept in mind that the elicitation interview is a means to obtain descriptions of experiences that are subjective and private in nature. We collect the "best way" mathematicians can describe what happened during their intuitive experiences. The fact that mathematicians feel they depart from the outer/physical world cannot necessarily be seen as an indication of the existence of a Platonist or constructivist world of mathematics. Indeed, describing the acts and procedures that occur in our mind is a linguistic achievement; as Findlay (1948) noted, there is a lack of suitable words to clothe such mental gestures and "events". The frequent use of "*as if*" or "*it's like*" in the mathematicians' narration indicates a description by analogies that implies likeness and difference. As reported earlier, the mathematicians' hesitations, comings and goings, and long silences during the re-enactment revealed the balance they sought between likeness and difference in an effort to provide us with an acceptable (satisfying) description that fulfilled their inner criteria of faithfulness[15].

The outcome of the interviews enabled us to access detailed descriptions of *being connected* to her/his discipline and to a particular mathematical problem. In order to gather descriptions of this experience, we asked the mathematicians to re-enact and describe "*what happened*" after the link was established, and whether it was a voluntary or an involuntary gesture. From the analysis of the texts, two main characteristics came forth:

- There is an apparent passive attitude regarding mathematical mental activity. One mathematician describes the preparatory phase as a moment of contemplation; once the link is made, the receptive mathematician allows herself /himself to "*contemplate*" mathematics. For example: *"It's not a guided thinking, it's like I let ideas appear*"

[14] The lack of proper/satisfying words (according to mathematician's opinion) to describe what happens when the subject is "connected to math" led us, in some cases, to give up on the elicitation of that particular moment as the frustration of the interviewee (that can be caused by her/his difficulties to find a description acceptable for her/him), could be damaging for the pursuit of the EI.

[15] On the difficulty of obtaining a description with the elicitation method, the reader can find a detailed discussion in Vermersch (2014), where the author carefully assesses the characteristics of what a description is and what horizon a phenomenologist must have when seeking a description of a singular lived experience.

(a10) – "*I don't know if we can call this 'to think', it's not under the control of will*" (a11).

- Mathematicians describe their experiences as if they were spectators to a mathematical *play* (we borrow the term from one of the interviewees): "*It's rather to see, it's like equations, things written on little pieces of paper*" (a12); "*I see formulas that assemble together*" (a13); "*A rotating apparition of different images and memories*" (a14); "*These ideas that gravitate one around the other*" (a15).

b- The signs of portent of the insight

The analysis of the transcriptions did not allow for a straightforward discrimination between the preparative period and what would be considered the moment just before insight (Item *i* of the three-step sequence[16]). While the EIs were carried out, the reflective content and (even more so) the pre-reflective content occurring during the moment before the surge of the insight was often felt by the interviewee as fuzzy, knotted, "mysterious" and introspectively opaque[17]. Therefore, the descriptions of the acts, gestures and feelings characterizing this singular moment were in some cases difficult to access. As such, the following criterion was used to distinguish what was to be attributed to one phase or to another: the moment *just before* is characterized by the "detection" of signs of portent that lead to a change in the dynamics of the experience. While during the preparatory phase, the subject is in a state where she is *broadly* connected to mathematics[18], a modification of the mathematician's attention occurs at the moment that precedes the surge of insight. This alteration occurs as the subject becomes aware of a mathematical matter that catches her attention and that she considers as noticeable. To employ a scientific metaphor, there is a cessation of the "scanning" of mathematical landscapes and the mathematician's mind ceases to wander.

Whatever the connection to mathematics evidenced during the preparative period (voluntary and involuntary), the descriptions of the first step of the "*i-ii-iii*" sequence introduced at the beginning of this section converged towards the following: the moment preceding the surge of insight involves the mathematician perceiving a particular mathematical idea or statement that lasts, that somehow differs from the others or "behaves" differently. The mathematician's attention is captured by "*something*".

For example:

-"*There is that idea that arrives, here…*" (b1);

- "*Voilà! There is something*" (b2);

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[16] Item *i*: the moment where signs of portent are perceived.

[17] This is an impression that almost everyone unfamiliar with introspection, and even more with elicitation interviewing, harbors at the start of the interview. As the interview goes on, the interviewee gradually becomes aware of the pre-reflective gestures of her/his experience. Nevertheless, a limit in the discrimination of these gestures is often reached.

[18] Recall, "*I'm visiting mathematical landscapes*" (a9). We give additional excerpts of the speeches that we did not include in Subsection *a* so as not to flood it with quotes: "*Technically, I'm not paying attention to anything precise*" – "*I let myself contemplate*" – "*I start perceiving things*".

-"*There is one idea that stays, I try to keep it*" (b3);

-"*When I see the phrase X, it carries a lot of weight*" (b4).

c- <u>The insight's surge</u>

In each case, mathematicians had a poor reflective awareness of what was related to the micro-dynamics of the insight surge. Therefore, the pre-reflective content of this particular moment was the most difficult to obtain. Firstly, due to its suddenness and briefness, expressions like "*instantaneous,*" "*immediate*" and "*a flash*" were collected. Secondly, the difficulty in accessing the description of the surge lies in the fact that this particular moment appears to be highly pre-discursive (i.e.: "*I had the impression*" (c1) – "*We have sensations, feelings that come to the mind*" (c2) – "*It's more I'm feeling things, I cannot put words on them*" (c3)). However, the very briefness of that instant and its discontinuous feature, as felt by the subject, were the characteristics of the insight surge and thus provided an easy criterion to identify it during the EIs. Two dominant qualities were extracted for the description of this central moment:

i) The instantaneity of the moment of insight is the first characteristic common to the intuitive experiences investigated. Words like "*immediate precipitation*", "*instantaneous flash*" or "*all of a sudden*" are employed to describe it. The insight delimits a *before* and an *after* that is discontinuous – at least at the level of pinpoint description reached during the EIs;

ii) the second characteristic is the absence of any particular act (or gesture), made by the mathematician, that may cause the surge of the knowing or the proposition that seems true. The surge of the illumination is described as the very moment where there is a complete void and where "*if there is something special in it, it is that nothing is happening*" (c4). The absence of a particular act or gesture at the moment of insight (or at least any perception of such) is acknowledged by a mathematician as being akin to "*an effortless knowing* [that] *just pops up*" (c5).

Although the moment of the insight seems to be devoid of acts, efforts to more accurately describe the sensations associated with this moment reveal something we have stated earlier, in the section devoted to the preparative phase: for most mathematicians, the passive character of the access to the illumination is noticeable. For example:

- "*It's like if all of a sudden this atmosphere imposes itself in front of me*" (c6);

- *"I'm starting to see something. It must be caught because it's not that logical, it has no anchor"* (c7);

- "*It like precipitates immediately and then you see it – it just kind of pops up*" (c8);

- "*Everything started to fall into place*" (c9);

- "*There is this idea that comes, here, and suddenly in respect to this idea all the existent contradictions collapse*" (c10); "*it's like all these ideas that gravitate around each other start collapsing. At the end the repulsing forces between them disappear, at that moment everything imbricates*" (c10-bis);

- “*It must organize by itself*” (c11).

Here the quasi-spectator attitude of the mathematician is again noticeable: the appearance of insight is described as an outer process that the mathematician cannot force or provoke, rather than the result of a mathematician’s discursive activity or of a construction (i.e.: “*it must organize by itself*” (c11); “*It’s like if all of a sudden this atmosphere imposes itself in front of me*” (c12)).

In order to prevent any misinterpretation of the foregoing points, we want to underline that by showing the passive (or spectator-like) aspect of mathematician’s attitude in the insight arousal, we do not conclude that the insight is reducible to a posture of total passivity for the mathematician. As the illumination arises, the mathematician is in a state of receptivity that allows her to perceive the insight. However, this penetrative moment is a consequence of intense conscious preparative activities – as evidenced by the portent signs of the insight – that prepare the ground for intuition. Let us also note that our method of investigation does not claim to conclusively determine that the preparative phase generates an intense unconscious activity, though this possibility can’t be excluded.

d- Becoming mindful of the insight.

The descriptions of the pre-reflective content point to two classes of features related to the criteria according to which mathematicians recognize the insight *as such* (i.e criteria for truth and criteria for the grounding of knowledge). This was achieved through the effectiveness of the EI method at elucidating the factors leading to a judgment. Indeed, one of the benefits of the EI protocol is that when a subject expresses a judgment, the interviewer immediately asks which criterion the subject has used to arrive at such a judgment. For instance, if the subject says she/he is happy, the interviewer asks “*how do you know that you are happy?*” in order to access the internal subjective criteria that allow the subject to perceive her/his happiness. Hence, we investigated what were the mathematicians’ criteria used to apprehend the insight as an insight (*e.g.* “*how do you know that this is the solution?*”). In other words, how did the mathematicians know that the outcome of the insight was a true mathematical statement[19]. This point was of particular importance as, immediately after the illumination, the mathematicians acquired an unquestioning certainty *(“I was convinced, this conviction was absolute, immediate”* (d1) - *“You just kind of know and it’s clear, very absolute”* (d2) - *“I’m sure it’s going to work”* (d3)). We brought to light the criteria used for the recognition of the insight and list them below:

- Achievement of completeness: *“It is a completeness of the arguments”* (d4)*, “An unassembled puzzle: an extra piece is put and the complete image pops up, although numerous pieces are still missing”* (d5);

- Higher degree of generality of mathematical knowledge: *“I’m sure it’s true because it’s not the same level of generality”* (d6);

- Order: *“Everything started to fall into place”*(d7)*, “Everything imbricates by itself” (d8), “It imbricates well in our mental structure”*(d9);

- Simplicity: *“Suddenly things become much simpler!”* (d10).

[19] We speak here about subjective criteria that are not straightforward justifying checks. We return to the latter aspect later in the text.

Another class of subjective criteria was evidenced: a characteristic of the insight is that it gives access to a global vision of mathematics. The outcome of the insight is the perception of a general architecture, of a whole image that shows "*where to go*". Indeed, once recognized, the outcome acts as a guide (we collected the expression "*We know how to work afterwards…*") for the justificatory activity subsequently needed in order to establish the insight as a mathematical truth, or to fill up the gap above which the mathematician has jumped thanks to the illumination arousal. Here are pieces of the descriptions of such subjective intuitive experiences: the mathematician is interested in *"How to organize things: we don't focus on something in particular"* and with the insight "*A principle, a general architecture (of the proof) has been brought out: we know from where we come and where we go. The feeling of chance has vanished*" (d11)[20]. The illumination is acknowledged through the feelings of coherence and consistence that are brought to the mathematician's mind; the illumination makes the mathematician aware of pieces of mathematics at a larger scale, where details are not what matters – at least at the instant of insight.

Intuition is a knowledge-grounding experience by which the mathematician accesses knowing beyond evident rational inferences and perceives an indivisible piece of mathematics that surges as a whole and does not show evident proof-making steps. For example: "[at this moment] *I realized that I had no need for a proof to be convinced of it"* (d12) and "*It's not that I have the proof but* [I know it's true] *because all the arguments that make it true are here*" (d13).

In one significant (but not necessarily representative) case, the moment following the advent of the insight was described as a fragile instant of certitude at the moment when the mathematician becomes aware of it. The intuitive knowing is to be "*firmly caught as it is not that logical and has no anchor*" (c7) because "*the thing is here but it is not fixed, it is not here for ever*" (d14). This description was of great importance for it evidenced that to anchor it, the mathematician proceeds to run little tests to check if it is stable: "*I make it move, it* [the insight outcome] *is there, I try to push it a bit on one side and another… to check if it is really there*" (d15).

Finally we would like to report on the bodily and mental feelings/sensations that are experienced by the subjects. The attributes of the subject's inner states once the insight is seen/grasped/perceived that are listed below were present in all the experiences we investigated, though in various intensities depending on the mathematician. They are related to:

- An irreducible sensation of ease: "*A relaxing of my focus, like I can rest for a second*" (d16);

- Pleasure: "*It's a moment of narcissistic pleasure*" (d17) – "*I savor*" (d18);

- Security: "*A mental sensation of security, and there is this bodily sensation of "Phew!", of "It's going to be OK*" (d19);

- Comfort: "*You feel comfortable, it's a bodily feeling of comfort*" (d20).

Moreover, the verbalization of this synchronic dimension of the experience permitted the interviewer to check on the effectiveness of the re-enactment. Indeed, alterations

[20] Recall also the description of the mathematician that sees the whole picture of the not-yet assembled puzzle – quoted in Subsection c.

in the expressions (pleasure and relief) of the interviewees and, at some point, the physical re-enactment of the experience (modifications of the posture, noticeable alterations in the way of breathing…) were external signs of the subject being in an evocating state.

D- What does the intersubjective structure of the moment of insight tell us?

As discussed in the previous sections, a phenomenological approach was used to investigate the mathematical insight experience. Being phenomenological, our study shows three main characteristics: it is descriptive, it uses the phenomenological reduction, and it searches for the essence of the insight experience. Hence, an inductive perspective was adopted as we started with descriptions of genuine lived experiences involving mathematical insight in the aim of subsequently disclosing the invariant structure of this phenomenon. The methodology employed to analyze the raw descriptions (detailed in Section B) permitted the identification of an essential structure composed of four constituents. The collection and subsequent analysis of the subjective data were carried out within the phenomenological reduction. Stated differently, we wanted first to access descriptions of a phenomenon (the surge of an insight in mathematical research) *as effectively lived* by existing subjects (contemporary mathematicians) to give an account of the manner in which this phenomenon appeared to *individuated* consciousness. We would consider the descriptions as they appeared, without adding to or subtracting from them. Then, and only then, could we conduct an analysis and synthesis of the data with the aim of revealing the essential structure of the intuitive experience in mathematics, *i.e.* a generic structure[21] that reflects the essential properties of the intuitive experience in mathematics and its conditions of possibility.

The specificity of the approach was to go beyond testimonies and spontaneous verbal reports that may have narrated the circumstances and the content of the surge of an insight in mathematical practice. We handled subjective data from the reflective side of genuine lived experiences that was complemented by the pre-reflective content of those experiences, revealed through a specific protocol of guided introspection. Indeed, illumination and insight often surge suddenly, *apparently* without particular reason and, as a result, the reflective component of such experiences is often poor and scarce. Hence, to arrive at a meticulous description of the micro-dynamics of this particular phenomenon, it was necessary to look carefully at the pre-reflective content of these singular experiences and zoom-in on the experience of a particular temporal moment (or time-slice) during the intuition arousal.

When experiences of insight in mathematics are magnified thanks to the EI method, and their cognitive and mental features thus revealed, we find that the insight surge is much more than the discontinuity of its upsurge and that is a phenomenon that shows specific and distinct moments. It is a process in which a preparatory activity is needed - a disconnection from the "out-here" that permits an intimate connection to mathematics - and that is included in a particular sequence in which the mathematician becomes available to the arousal of the insight and recognizes it as such - what we labeled as "*becoming mindful of the insight*". As opposed to intentional, directed or inferential thinking, the insight surge is a process that cannot be forced. However, it is *not a spontaneous impression*, nor a *revelation of implicit knowledge*. Indeed, we evi-

[21] In the sense of *trans-experiential* – the structure that transcends the individuated structures.

denced that the mathematician enters a state of receptiveness that makes her able to *perceive* the insight (note: this is likely a *one way* inference, as it does not imply that each time the mathematician puts herself in a contemplative or abandoned attitude, an insight will surge[22]). This receptiveness goes along with a passive attitude of the subject: the mathematician achieves a non-conceptual knowing in an instant during which no act seems to be perceived and where it feels to her as if something was happening *to* her instead of her doing it. The intuitive experience in mathematics is an experience in which the mathematician grasps, after an intense preparatory activity[23], a truth and recognizes it.

Before discussing the philosophical implications of the generic structure that we unfolded, we would like to underline that the collection of the mere reflective contents of the different experiences *would not have permitted* the identification of the essential structural moments of the intuitive experience. Indeed, before the interviews, many aspects and gestures of the experiences where not (yet) reflectively available in the participants' consciousness. The use of the EI method to perform the assisted introspection was thus ideal for uncovering the pre-reflective components of the experiences and was absolutely necessary for the achievement of our goal. The guided introspection was the means for 'unfolding the immediacy' of the insight upsurge. Collecting the descriptions of the mere circumstances and the subjective reports on what mathematicians 'thought' they had lived would not have sufficed. Re-enactments of the experiences were necessary.

As said in the introduction, mathematics is not reducible to the processes of inferential and discursive thinking. There is a great deal of literature arguing that intuition is a cornerstone of mathematical advances (*Poincaré, 1908; Hadamard, 1945; Dieudonné, 1975; Govers, 2002; Celucci, 2006; Hersh, 2013*). For instance:

"*La logique qui peut seule donner la certitude est l'instrument de la démonstration ; l'intuition est l'instrument de l'invention*" (Poincaré, 1908) (*Logics, that only provides the certainty, is the instrument of proof, while intuition is the instrument of invention*).

"*The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there never was any other object for it*" (Hadamard, 1945).

"*La qualité essentielle d'un mathématicien est l'imagination : la logique ne sert qu'à mettre les démonstrations sous une forme irréfutable, elle est incapable de les suggérer*" (Dieudonné, 1975).

A meta-analysis of selected writings of mathematicians was detailed in two recent articles, in which the intuitive aspect of mathematical practice was investigated by putting in perspective first-person written reports of contemporary researchers in mathematics and the extensive literature on mathematical intuition (Author, 2015a; Author, 2015b). The outcome of the present investigation acknowledges and enriches previous conclusions and conceptions of creative mathematical practice: the insight is

[22] On the role of contemplation and meditation on mathematical progresses and teaching, see (Wolcott, 2013).

[23] This preparatory work might not necessarily be done just before the intuition surge. We collected testimonies where the intense preparatory work was an accumulation of work, made over hours, weeks or even months.

indeed fully constitutive of mathematical advances and progresses. Mathematics seems not to be reducible to logic and pure reasoning and intuition in mathematics shows essential features that can account for its founding role. At the same time, our study shows that the intuitive experience in mathematics has "*nothing to do with the ancient mystical myth of an intuition that would surpass logic by making a direct connection to the Transcende*ntal" (Hersh's wording in (Hersh, 2013)). Intuition is not spontaneous but requires time and has 'conditions of possibility'.

On the ability of mathematical intuition to grasp a truth and entertain it, Giommi and Barendengt (2014) proposed a parallel between intuition in mindfulness meditation and mathematics. The two authors acknowledged mathematical intuition as being, like insight meditation, an experience that *is a comprehending, an act – as opposed to a conceptual content – and an evidence that manifests itself immediately as indubitable and unquestionable* (Giommi & Barendegt, 2014*)*. Later, this evidence is to be translated into words and conceptualized; what is missing (in our case the proof of the insight veracity) will be sought. Hence an intuition cannot be described as a bottom-up process but like "*a real presence in the human mind of the possibility of a higher form of knowing, that is, in its essence and mode of operating, different from inferential, rational, discursive thinking*", where intuitive knowledge "*is primarily concerned with being able to acknowledge what is manifesting in our experience moment by moment*" (Giommi & Barendegt, 2014 – p.2)

The reflective and pre-reflective content of the investigated intuitive experiences emphasize the mathematician's ability to entertain what is manifesting to her in certain moments of activity; access to the insight appears devoid of effort (to the point of the subject passive attitude evidenced in some interviews). The consequence of this experience is a comprehending, a sudden approach to a clear vision or to a mindfulness of a global picture of "what is it all about"[24]. The outcome of our study gives to mathematics a Platonic facet for, in the lived experiences we elicited, mathematicians seem to "dis-cover" a mathematical reality yet existing (see also Author, 2015a).

The particular aspect of *unquestionability* that appears to be a common feature of mindful meditation and mathematics (Giommi & Barendegt, 2014*)* is in fact brought out by what was obtained through EIs. The sensation of certainty, along with the feeling of security (that can even be physical), are direct consequences of the insight: through this flash and sudden illumination, mathematics turn into a *whole* that is clear, ordered, simple and complete. Under the intuition illumination the mathematician happens to perceive things under a different perspective, actually under the *right* perspective. This new perspective unifies facts/propositions that seemed previously motley and shows the picture that lies under the unassembled puzzle. Noticeably, the notion of right perspective echoes with 20th century mathematician A. Grothendieck's analysis of his own mathematical practice in *Récoltes et semailles* (Grothendieck, 1986). According to Grothendieck, "fertile points of view" are fundamental - if not inescapable - for sounding mathematical advances. Indeed, such a point of view is the very one which "*is that eye that at the same time makes us discover and recognize the unity in the multiplicity of what is discovered*" (Grothendieck, 1986 – pp. 40-44). This notion correlates with the features of mathematical intuition investigated in our study,

[24] On the latter expression, the "all" might refer only to a part of a wider picture but still, the mathematician suddenly has access to a sharp vision of a piece of mathematics.

and was indeed pointed out by the participants (*e.g.* "*it [the insight outcome] unifies a lot of things*").

As uncovered, the intuitive experience in mathematics is an act of perception that leads to a finding; it is a perceptive experience that allows the mathematician to access a fact, a statement or a property that is sensed as true through a non-deductive process in which both meaning and comprehending are involved. The experience of intuition in mathematics has aspects common with the experience of sensory perception, with the notable distinction being that what is perceived is abstract matter rather than concrete matter. Chudnoff (2011; 2014) describes this as a *perceptualist* view of intuition in mathematics[25]. Chudnoff's *perceptualist* view of mathematical intuition acknowledges the existence of experiences where *i)* an abstract subject is present to the mind *not merely* as a concrete illustration of that abstract matter and *ii) seeming* does not simply derive from illustration (sensory representations of the mathematical matter in question[26]) but is grounded in thoughts (see Chudnoff, 2014, pp. 184-186 and the references herein).

The intersubjective structure we identified resonates significantly with Chudnoff's conception of mathematical intuition. Indeed, during their own intuitive experiences, mathematicians do not necessarily access the insight content through illustration or imagery. In several cases, it was thoughts without imagery and matters of pure thinking that were involved in the experiences. For example: "*there is this idea that comes, here, and suddenly in respect to this idea all the existent contradictions collapse*" (c10) and "*it's like all these ideas that gravitate around each other start collapsing. At the end the repulsing forces between them disappear, at that moment everything imbricates*" (c10-bis). As a matter of fact, the mathematician does not necessarily "visualize" a particular mathematical object or inevitably have an illustrative perception of a mathematical proposition. "Becoming aware" of the insight is an intellectual act of perception in which the mathematician perceives the insight *and* something which is relevant to the insight's truth value, giving to the experience what we may call *fullness*. To borrow Chudnoff's terminology, for a mathematician, having intuition is being able - together with the insight perception - to entertain thoughts about the insight content that can prove it (i.e.: "*it is not that I have the proof for it but everything that makes it true is here*"). In the intuitive experience, the mathematician succeeds in attaining an awareness of a mathematical proposition that can be based only on thoughts. The insight is apprehended or recognized as such because "*whenever the mathematician has an intuition representing that P, her mathematical intuition also makes it seem to her as if she is intuitively aware of the items in virtue of which P is true, and it does so in virtue of making it seem to her as if she is in a state that enables demonstrative thoughts about those items*" (Chudnoff, 2014 – p.186).

[25] Chudnoff defines that view as follows: "(…) *mathematical intuition is a kind of experience that is like sensory perception in giving its subjects non-inferential access to a world of facts, but different from sensory perception in that the facts are about abstract mathematical objects rather than concrete material objects*" (Chudnoff, 2014 - p.174).

[26] Klein defended a mathematical intuition grounded in the imagination, specifically a visualization of concrete illustration of abstract matter (like points, lines...)

What we labeled the *fullness* of the intuitive experience reveals an important aspect of mathematical intuition[27]: it shows presentational phenomenology. For perceptual experiences presentational phenomenology can be described as such: *whenever one has a perceptual experience representing X – for instance that the cat is sleeping on the couch – one's perceptual experience also makes it seem as if one is sensorily aware of items in one's environment (like external spatio-temporal particulars) in virtue of which X is true – for instance, the cat sleeping on the couch* (Chudnoff, 2011b). In the case of intuitive experiences in mathematics, as *i)* for the great majority of experiences, intuition involves abstract matter and *ii)* since the insight does not surge all alone but with a "cortège of facts"[28] that seems to justify it, we can propose that "*whenever the mathematician has an intuition with respect to a proposition A, she both perceives that A and something relevant to A's justification"* – giving thus to the mathematician a presentational phenomenology with respect to A since the descriptions and characterization of intuitive experiences in mathematics can address its capacity for justifying beliefs and providing knowledge. Indeed, by evidencing these facets of intuition, we show that intuition in mathematics is not illusory or pre-scientific and does not consist of merely vague ideas. It is part of an intellectual activity where – if a certain inner state is attained through a preliminary training and professional expertise – the mathematician has access to a vision of a whole piece of mathematics (a certain *landscape*, as described by a mathematician) that will be translated at a later point into concepts[29].

We will now complete this discussion by turning to the philosophical implications that our study, while also considering how our results can be put in line with Husserl's theory of intuition. In "*Les phases décisives dans le développement de la philosophie de Husserl*" Biemel (1959) recalls Husserl's desire to emancipate ideal objects from psychology and demonstrate their independence, with the ultimate aim of addressing how they come to be given (Husserl, 1970; Husserl, 1983). Biemel considers a passage in the *Nachlass* that states:

"*when it is made evident that ideal objects despite the fact they are formed in consciousness, have their own being in themselves, there still remain the enormous task which has never been seriously viewed or taken up, namely the task of making this unique correlation between the ideal objects which belong to the sphere of pure logic and the subjective psychical experience conceived as a formative activity, a theme for investigation. When a psychical object as I performs certain (and surely not arbitrary but quite specifically structured) psychical activities in my own psychical life, the successive formation and production of meaning is enacted according to which the number-form in question, the truth in question, or the conclusion and proof in question emerges as the successively developing product*".

Biemel particularly notes the sentence concerning psychical activities ("*and surely not arbitrary but quite specifically structured*") to emphasize that the subject cannot arbitrarily constitute any meaning whatsoever, but that the psychical activities are the

[27] Recall that the experience of mathematical intuition corresponds to the experience of the insight in mathematical practice, meaning an experience where a proposition seems true.

[28] Chudnoff uses the term of *truth-maker* (Chudnoff, 2011b).

[29] Rodin writes, "*I shall stress that matching concepts with intuitions always was and, in my view, will always remain a principle purpose of mathematics*" (Rodin, 2010).

constitutive acts dependent upon the essence of the object considered. Put differently, the essence of a mathematical object does not depend on the psychical activities required to form that object. For instance, in order to understand the meaning of the number three "*we must perform determinate acts of collective connecting, otherwise the meaning of three will remain entirely closed to us. There is something like the number three for us when we can perform the collecting-unifying activity in which three comes to be able to be presented*" (Biemel, 1959) [30].

The author then clarifies the term of *constitution* and cites an enlightening excerpt of a letter sent from Husserl to Hocking in which the phenomenologist clarifies the meaning of the concept of constitution employed in the *Logical investigations*: "*The recurring expression that "objects are constituted" in an act always signifies the property of an act which makes the object present: not "constitution" in the usual sense*". Consequently, Biemel stresses the misunderstanding of the problem of constitution and denies that Husserl's intent was related to any kind of *production*. He proposes this central Husserlian concept to be understood as *restitution* as the subject restores what is already there. This, however, requires the performance of certain activities. We read: "*the best way to discuss the term of constitution is to discuss <u>the-becoming-present-of-an-object</u>. The acts which make this becoming-present possible, which set it in motion, are the constituting acts*".

We do not intend to enter into the details of the problem of constitution. However, we definitely want to acknowledge the importance of Biemel's proposition (defended as well by Zahavi (2003 – p.72). Earlier in this section, we pointed to the Platonist aspect of the intuitive experience in mathematics as it allows for the *dis*-covering of a mathematical truth. Put in perspective with Biemel's work, the generic structure of the intuitive experience correlates with it: intuition in mathematics is an experience of an *object-becoming-present* where the object is not produced but rather restituted. In other terms, the intuitive experience is the very one that makes an object present (Husserl's terminology) and during which the mathematician restores (Biemel's wording) what is already there – making this experience a founding one.

We should nevertheless remain humble, for our investigation does not fully address the constituting acts that the mathematician performs to make that becoming-present possible. However, the essential structural moments of the intuitive experience seem to reveal the independence of mathematical facts[31] and disclose the particular feature that these facts are intuitively given.

There is a last point that we want to discuss: the essential form of intuition introduced by Husserl in the *Ideas pertaining to a pure phenomenology and to a phenomenological philosophy* (Husserl, 1983) and that - in particular - concerns mathematical entities. This aspect of Husserl's phenomenology of intuitive knowledge may appear controversial for it is difficult *a priori* to see how the awareness of mathematical entities could qualify as intuitive. Husserl intended to justify such a claim in highlighting the analogies and differences of the essential form

[30] We borrow the English translation of Biemel's article made by J. Cogan.

[31] We prefer the use of *facts* rather than *objects* for the experiences we elicited were referring to the intuition of a mathematical statement or property and not to that of an object. An article on the particular issue of how mathematical objects are *effectively* perceived and *conceived* by mathematicians is in preparation. The philosophical impacts of the essential insight of mathematical objects apprehension will be presented thus there.

of intuition with straightforward perception, leading to the necessity of addressing two problems if we were to give a proper description of what essential intuition is. The first one concerns the sensuous representative content of essential intuition. The second one refers to the problem of how straightforward perception can provide any evidential foundation for essential intuition, since the objects of straightforward perception are concrete and contingent existences, while essences and ideal objects are not. Kidd (2014) has recently published a meticulous work on Husserl's phenomenological theory of intuition and proposed answers to these two issues. We make use of his clarifications to develop what follows.

The first point cannot be fully addressed at present as to complete this task the mathematical elements of the intuitive experiences should be considered in order *i)* to properly figure out the formative (mathematical) activity that leads to the perception of the insight and *ii)* to investigate the extent to which the (again mathematical) content of the intuitive experience reveals a process of ideation that intends to grasp structural features. Nonetheless, we may suggest that the presentational phenomenology of the intuitive experience in mathematics plays an important role. Quoting Kidd, "*like synthetic categorical intuition, ideation is a process of active synthesis that requires the subject to take up an intention to make explicit something only implicitly meant in the horizon of straightforward intuition. However, unlike synthetic intuition, more is required than to make explicit the formal features that are implicit in a single straightforward experience. We must also take up the intention to grasp the structural features of this object that are common with every object of a certain class"* (Kidd, 2014 – p.145), we may make the parallel with subjective descriptions of insight arousal. Indeed, the mathematician does not rest with the awareness of a single mathematical proposition or fact, for the upsurge displays a presentational phenomenology that provides her with mathematical presentations that seem to be consistent with the mathematical essence of the intuition content[32]. Were it not the case, the mathematician would simply deal with an 'impression' that would be devoid of 'epistemic warrants' (see Miscellaneous *vi*).

We believe that the second issue can be answered if we again consider the presentational phenomenology of the experience of intuition in mathematics. Indeed, presentational phenomenology makes ideal objects with concreteness and contingency thanks to the truth-makers (Chudnoff, 2011b) or the procession of facts that accompany the insight perception, giving the intuitive experience its foundational character. The generic structure, along with presentational phenomenology, show that mathematical intuition does not take place *out of the blue* and is not an immediate seeing/grasping, but rather a constitutive (in Biemel's sense) process[33] that allows the mathematical fact to be seen by the mathematician 'in its objectivity'.

E- Miscellaneous comments.

i) In spite of our interest in the context of discovery in mathematics beyond any justificatory activity performed by the mathematician, our investigation put forward

[32] As a verbatim example "*it is not that I have the proof for it but everything that makes it true is here*" or Grothendieck's phrasing "*it is that eye that at the same time makes us discover and recognize the unity in the multiplicity of what is discovered*" (Grothendieck, 1986).

[33] Kidd proposes the following answer to the second point: "*essential intuition is not an immediate seeing of the essence in one shot, as it were, but is instead more like what cognitive scientists today call 'pattern recognition'*" (Kidd, 2014 – p. 146).

the reasons and criteria according to which the mathematician recognizes the insight and gives it credence. Indeed, the mathematician perceives the insight and entertains it as knowledge grounding because, along with the insight, *prima facie* grounds for it are perceived. In other words, the grasping of insight is accompanied by justifying "sub-levels" - "Chudnoff's truth-makers" - indispensable for the insight to be valid at first sight[34], even if those do not involve inferential or rational thinking (at least to the level of detail reached during the elicitation interviews).

ii) Aside from its impressive and sound character, intuition in mathematics remains on the "discovery side" and is never considered proof of the validity of a mathematical statement. Mathematical intuition is fallible and subject to modification – as the results of essential intuition are (Husserl, 1970; Kidd, 2014). We investigated the intuitive experience for itself and gave an account of the structural moments of an experience that stays below the justificatory activity of proof writing. Interestingly, the two elicited intuitive experiences were related to moments of insight that occurred as the mathematicians were working on proofs. This seems to show that, even at the level of mathematical justificatory activity, discursiveness and rationality may not suffice.

iii) For each participant, the guided introspection allowed them to discover the micro-dynamics of their singular experience. It was a means for them to enter into their own intimate way of doing mathematics. This is important considering that during their daily research activity, their attention is mostly directed towards the content of their research (the *what* as opposed to the *how*). Prior to the interviews, the mathematicians were rather convinced that the insight had surged all of a sudden, leading in some cases to the strong belief that the description of the illumination advent would stay introspectively opaque.

iv) We were interested in submitting the generic structure of the insight surge to the participants in order to see if they would 'recognize their experience' in it. To a certain extent, we felt that general approval of the structure would secure the validity and value of our phenomenological approach. Indeed, if not carried out properly, our investigation would have led to a biased generic structure that bore little resemblance to the mathematicians' experiences. We proceeded thus to a systematic presentation of the generic structure to the participants and asked them to evaluate if this structure reflected 'the essence' of their singular experience. Each mathematician affirmed that it did indeed reflect her or his experience. However, we would like to underscore the following: we are not saying the participants were the ultimate "censors" or guarantors for the validation of the generic structure. Firstly, because mathematicians are not trained phenomenological researchers, we do not expect them to perform any level of eidetic phenomenological analysis – even of their own private experience! Secondly, the raw descriptions are given by subjects speaking from within the natural attitude and, as such, using a language that is concrete, detailed and specific to each individual subject. On the contrary, the language used for the generic structure is general and condensed. From the participant's point of view, the concision of the essential structure may be akin to a fleshless skeleton, lacking details and liveliness. However, these latter attributes are not what a phenomenological researcher looks for. The generic structure is articulated around key moments that play a specific role and that have a specific meaning from the researcher's perspective. This means that,

[34] The expression "*at first perception*" would actually suit better.

despite the adoption of the phenomenological reduction, this perspective inevitably colors the generic structure. As a consequence, the generic structure that is proposed cannot be presented as final or indisputable. As with any experiential and phenomenological research, an essential structure is subject to modification in light of further research or counterexamples (achieved through 'free variation') that may modify that structure. For a discussion on eidetic intuition, the reader can refer to (Sokolowski, 2000) and will find in (Giorgi, 1985 and Giorgi, 2009) thoughtful comments on the outcome of free imaginative variation as performed in the DPPM.

v) Finally, at the end of Section D we discussed the necessity of looking at the mathematical content of the intuitive experience in order to disclose – if present or identifiable – the sensuous content of mathematical intuition and the 'constituting acts' that lead to its perception. Though we have made a tentative account of this issue, continued close collaboration with mathematicians would be necessary to carefully investigate this point. Indeed, considering the advanced levels of mathematical skill involved in the intuitive experiences that were elicited, a scholarly delineation of the mathematical details and activities must be secured if the phenomenological researcher wishes to avoid misinterpreting or over-interpreting the subjective material.

Appendix

Below are two examples of EI transcription along with an illustration of how they were cleared out according to our methodology.

Example 1:

Here are pieces of unedited speech obtained using the EI method:

"It depends but let's say that in my case, there is really nothing precise. It seems that it would be an error to believe… <u>There were images that were appearing in my head</u> but they were not images of a logical series… It was rather a simultaneous vision… well not simultaneous but <u>a sort of rotation of different images and memories</u> about…

There are ideas, yes! That's it! <u>Ideas that come and go</u> – their number is roughly constant and they will come back one after the other but… well the question of knowing which one comes first does not make sense…

Actually no… In fact, <u>ideas are growing in number because more and more memories come to my mind</u>… I think I can associate this to "I must organize something but to what must I think about?". There are all these things that come to my mind but nothing is accurate. <u>I'm interested in organizing things, in finding priorities</u>…"

Reviewing these quotes reveals several aspects can be identified:

i) Comments: "*It seems that it would be an error*"; "*well the question of knowing which one comes first does not make sense*" that the interviewee spontaneously made to himself and that are just irrelevant to our purpose

ii) Retrospective interpretation of the experience: "*I think I can associate this to…*"

iii) A description that is not straightforward to identify as effectively being part of the experience or as again a retrospective evaluation of the experience. "*but they were not images of a logical series*". In doubt, such "descriptive items" are not considered

iv) Actual descriptions of the experience: they are underlined in the text and were kept for the analysis.

Example 2:

Below are two passages from a particularly successful interview, meaning that the unfolding of the micro-dynamics of the intuitive experience could be assessed in detail. For the excerpts given below, the researcher had noted that the mathematician's speed of speech was particularly slow, indicating that the interviewee was in close contact with the past experience and, as a result, was actually describing his mental gestures and was scarcely making comments on it (Petitmengin, 2001; Vermersch, 1999).

"*I'm just trying to recollect in my head (mind), in an abstract manner, what were the fundamental parts and how I can adjust them if I only have this little result… Well… (hesitating, long silence) there are those fundamental steps in the proof and there are ideas, like steps of a staircase… and thus I am trying to recollect the fundamental steps, the important ones that lead to the result.*

And then I am trying to mentally adjust things and to state that there is a step that… well the aim of the proof was to be able to generate as much matrices as possible… And here we were producing a lot! And here I am realizing that for a certain matrix family, there is only one that will generate other ones. So I am recalling exactly what is the proof step that I must adjust and how I must adjust it. But you know, I'm a bit disappointed because this result is one order of magnitude lower than what I was expecting".

We have underlined what was kept as descriptive items for the sake of faithfulness. As the mathematical details involved in these items are quite simple, we left them the way they were detailed.

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