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Research Statement
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The majority of my research is at the intersection of algebraic topology, category theory, and homological algebra. My working hypothesis is: *mathematicians create perspectives, and discover consequences.* My research process centers on a conscious effort to balance the visionary and the technical.

There are three main directions in my research.

1. Derived categories and the Bousfield lattice. A map of commutative rings $f : R \to S$ induces an adjoint pair of functors between unbounded derived categories $D(R)$ and $D(S)$. My research has looked at structure preserved under these functors, and consequently at the relationship between the localizing subcategories, thick subcategories of compact objects, and Bousfield lattices of $D(R)$ and $D(S)$. Furthermore, I've investigated the Bousfield lattice of several non-Noetherian rings, in the spirit of [DP08]. For example, I showed that in this context, unlike in the Noetherian case, there are objects that are not Bousfield-equivalent to any module [Wol11c].

2. Cohomological Bousfield classes and localizing subcategories. Cohomological Bousfield classes (CBCs) offer an approach to understanding and possibly classifying localizing subcategories in stable homotopy categories [HPS97, Hov95], or more generally in tensor-triangulated categories [CGR11, BCMR11]. In the category of spectra, one uses the Brown-Comenetz functor to show that every (homological) Bousfield class (HBC) is also a CBC; in particular, $\langle X \rangle = \langle (IX)^* \rangle$. I have done work extending the Brown-Comenetz construction to other categories, and constructing poset adjoints between the HBCs and CBCs. There are many open set theoretical questions in this area, and I have begun work with several set theorists to investigate whether answers depend on large-cardinal axioms.

3. Metamathematics and contemplative pedagogy. Metamathematics is the interdisciplinary study of the mathematical experience. Throughout my math graduate studies I have maintained a parallel interest in metamathematical literature - from cognitive science, to philosophy, psychology, and anthropology. I published a collection of introspective essays on the subject [Wol09], ran a reading group/seminar for two years, and most recently am building an online database of carefully documented experience [Wol11a].

I will address each one of these separately and outline the following: a brief background, my research contributions to date, and future directions. While discussing future research, I mention some possibilities for engaging undergraduates in research.

1. Derived categories and the Bousfield lattice

1.1. **Background.** The derived category $D(R)$ of a commutative ring R, from homological algebra, and the stable homotopy category $\mathcal{S}$ of spectra, from algebraic topology, bear many structural similarities. In the language of [HPS97], both are *monogenic stable homotopy categories*: they are triangulated with a compatible symmetric monoidal tensor product; the unit of the tensor is a compact, weak generator; arbitrary coproducts exist; and Brown representability holds.

Since the 1990s, an axiomatic approach has been used to translate questions and results among algebraic topology, algebra, modular representation theory, and category theory.

One main example is subcategory classification. A triangulated subcategory is called *thick* if it is closed under retracts, and *localizing* if it is closed under arbitrary coproducts. The groundbreaking [HS98] classified thick subcategories of compact objects in $\mathcal{S}$. Neeman [Nee92] successfully classified thick and localizing subcategories in the derived category of a Noetherian ring. Thomason [Tho97] extended this, classifying thick subcategories of compact objects in the derived category of a general commutative ring. Benson, Carlson, and Rickard [BCR97] established this classification in the stable module category $\mathrm{StMod}(kG)$, for a finite p-group G.

Another translated concept is the Bousfield lattice. The *Bousfield class* of an object X in a tensor triangulated category is defined to be

$$\langle X\rangle = \{W \mid X \wedge W = 0\},$$

where $- \wedge -$ denotes the tensor product. Two objects X and Y are *Bousfield equivalent* if $\langle X\rangle = \langle Y\rangle$. This is an equivalence relation, with a partial order given by reverse inclusion. When there is a set of such classes, we get a complete lattice, called the *Bousfield lattice*.

The Bousfield lattice was first studied in $\mathcal{S}$ [Bou79a, Bou79b, HP99]. There it is motivated by the fact that $\langle X\rangle = \langle Y\rangle$ if and only if X_* and Y_* homology have the same acyclics. In [HPS97] many results and constructions were generalized to other axiomatic stable homotopy categories.

The question of when there is a set (as opposed to a proper class) of Bousfield classes has received much attention. Ohkawa [Ohk89] showed this for $\mathcal{S}$, and Dwyer and Palmieri [DP01] showed this for Brown categories. Most recently Iyengar and Krause [IK11] showed that in any well generated tensor triangulated category, there is a set of Bousfield classes.

1.2. **My contributions.** Given a map $f : R \to S$ of commutative rings, we have the extension by scalars functor $f_* : \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}S$, sending $M \mapsto M \otimes_R S$. It has a right adjoint $i_* : \mathsf{Mod}\text{-}S \to \mathsf{Mod}\text{-}R$, the forgetful functor. These give functors between the chain complex categories $Ch(R)$ and $Ch(S)$, and also left derived functors $f_\bullet = L(f_*) : D(R) \to D(S)$ and $i_\bullet = L(i_*) : D(S) \to D(R)$.

In [Wol11d], I look at how $f_\bullet$ and $i_\bullet$ relate the subcategories and Bousfield lattices of $D(R)$ and $D(S)$, for various conditions on R and S and the map f. The subcategory classifications of $D(R)$ are based on the prime spectrum $\mathsf{Spec}\, R$, and use the notion of support. Furthermore, f induces a map $f^{-1} : \mathsf{Spec}\, S \to \mathsf{Spec}\, R$.

For example, I showed that when $f : R \to S$ is surjective and R and S are both Noetherian, the following diagram commutes.

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \text{arbitrary subsets} \\ \text{of } \mathsf{Spec}\, R \end{array} \right\} & \underset{(f^{-1})^{-1}}{\overset{f^{-1}}{\rightleftarrows}} & \left\{ \begin{array}{c} \text{arbitrary subsets} \\ \text{of } \mathsf{Spec}\, S \end{array} \right\} \\
{\scriptstyle supp(-)} \upharpoonleft\downharpoonright & & {\scriptstyle supp(-)} \upharpoonleft\downharpoonright \\
\left\{ \begin{array}{c} \text{localizing subcategories} \\ \text{in } D(R) \end{array} \right\} & \underset{i_\bullet}{\overset{f_\bullet}{\rightleftarrows}} & \left\{ \begin{array}{c} \text{localizing subcategories} \\ \text{in } D(S) \end{array} \right\}
\end{array}$$

Furthermore, I show that $f_\bullet$ and $i_\bullet$ give well-defined operations between the Bousfield lattices of $D(R)$ and $D(S)$, and preserve the well-studied sublattices DL and BA.

The most interesting case is when we drop the Noetherian assumption on R and/or S. This allows us to gain new information about the subcategories and Bousfield lattice of the derived category of some non-Noetherian rings.

In other work, I have looked at the derived categories of some specific non-Noetherian rings. Fix $n_i > 1$ and a prime p, and define

$$\Gamma := \frac{\mathbb{Z}_{(p)}[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)} \text{ and } \Lambda := \frac{\mathbb{F}_p[x_1, x_2, ...]}{(x_1^{n_1}, x_2^{n_2}, ...)}.$$

The derived category $D(\Lambda)$ is studied extensively in [DP08]. In unpublished work, I've generalized or extended many results from that paper, to $D(\Gamma)$.

In [Wol11c], I show that there are objects in $D(\Lambda)$ that are not Bousfield equivalent to any module, answering a question posed in [DP08]. This is in contrast to the Noetherian case, in which an object X is Bousfield equivalent to the module

$$\bigoplus_{\mathfrak{p} \in \text{supp}(X)} \overline{k_\mathfrak{p}},$$

where $\overline{k_\mathfrak{p}}$ is the image in $D(R)$ of the residue field $k_\mathfrak{p}$ of $\mathfrak{p}$.

1.3. **Future directions.** Every non-Noetherian ring is the colimit of its finitely generated submodules. The functorial properties demonstrated in [Wol11d] give me hope that there may be a way of looking at a category of derived categories, or a category of Bousfield lattices, and working there to get more of an understanding of the derived category of non-Noetherian rings. There are also many other examples of non-Noetherian rings to look at. For example, coherent rings are non-Noetherian but bear many similarities to Noetherian rings. Coherent rings show up often in non-commutative algebraic geometry.

There are examples of ring spectra R in $\mathcal{S}$ such that the homotopy category of R-modules is known to be equivalent to the derived category of the coefficient ring $\pi_*(R)$, see [Pat11]. Since the coefficient rings of many important ring spectra are non-Noetherian, this suggests the possibility of future application of my work to stable homotopy theory.

Undergraduate research possibilities. Homological algebra is first-year graduate material, but the axiomatic approach to the derived category makes it accessible to undergraduates. Furthermore, applying the Bousfield equivalence relation yields a poset lattice, which is easily accessible. There are many open questions and computations about derived categories of specific rings and their Bousfield lattices. Answers to these would help fill out our understanding, which at present is quite global and unrefined. I think there are opportunities for undergraduates to contribute, and in the process learn about rings, homological algebra, and lattice theory.

2. Cohomological Bousfield classes and localizing subcategories

2.1. **Background.** In the category of spectra $\mathcal{S}$, every object E defines a homology theory E_*, and every homology theory arises in this way. The (homological) Bousfield class (HBC) of an object is precisely its acyclics:

$$\langle E\rangle = \{X \mid X \wedge E = 0\} = \{X \mid E_*(X) = 0\}.$$

Similarly, in any triangulated category in which Brown representability holds, there is a one-to-one correspondence between objects E and cohomology theories E^*. So we define the cohomological Bousfield class (CBC) of E to be

$$\langle E^*\rangle = \{X \mid E^*(X) = 0\} = \{X \mid [X, E]_* = 0\}.$$

The collection of CBCs has been studied less than the HBCs that make up the Bousfield lattice. One reason is that, while homological localizations exist in most examples ($\mathcal{S}$ and $D(R)$, in particular), it is unknown whether cohomological localizations exist. Some recent work [CGR11, BCMR11] shows that the answer may depend on large-cardinal axioms of set theory.

Every CBC is a localizing subcategory. Hovey showed [Hov95] that in $\mathcal{S}$, every HBC is actually a CBC. There are pathological examples of tensor-triangulated categories in which the inclusion $\{\text{HBCs}\} \subset \{\text{CBCs}\}$ is proper. Thus CBCs may be a better context in which to attempt a classification of localizing subcategories.

2.2. **My contributions.** Hovey's proof that every HBC in $\mathcal{S}$ is a CBC uses the Brown-Comenetz duality functor $I(-)$. In particular, $\langle X\rangle = \langle (IX)^*\rangle$. In unpublished work, I've attempted to axiomatize the Brown-Comenetz construction, so as to extend Hovey's result. For example, this can be done in the derived categories $D(\Lambda)$ and $D(\Gamma)$ discussed above. Furthermore, in the derived category of a Noetherian ring (more generally, in any Noetherian stable homotopy category), this extension is possible, and it results in a proof that in fact $\langle X\rangle = \langle X^*\rangle$ for all X.

With the goal of understanding the relationship between the collections of HBCs and CBCs, I've constructed (under the assumption that there is a *set* of CBCs) a product map

$$\{\text{HBCs}\} \times \{\text{CBCs}\} \to \{\text{CBCs}\}$$

and adjoint poset maps

$$\mathcal{C} : \{\text{HBCs}\} \to \{\text{CBCs}\} \text{ and } \mathcal{H} : \{\text{CBCs}\} \to \{\text{HBCs}\}.$$

With respect to the inclusion of {HBCs} into {CBCs}, these reduce to well-known operations.

Set theory plays an important role in questions about CBCs and localizing subcategories. For example, in [CGR11], the authors show that cohomological localizations exist in any stable combinatorial model category (such as $D(R)$ or $\mathcal{S}$), assuming a large-cardinal axiom, Vopěnka's principle. The question of whether there is a set (as opposed to a proper class) of localizing subcategories, or of CBCs, is not peripheral – many constructions and results rely on having a set.

At a workshop on Large Cardinal Methods in Homotopy, at the University of Barcelona, I gave a talk [Wol11b] outlining the relevant set theory questions in homotopy theory. I also helped facilitate a problem session, attempting to bridge the gap between the set theorists and homotopy theorists. Specifically, we made progress in generalizing Bousfield's construction [Bou79a] of an operation $a(-)$ that gives

$$\langle X \rangle = \mathsf{loc}(aX).$$

Bousfield uses the notion of stable cells in $\mathcal{S}$ as a measure of size, and defines aX to be the coproduct of all X-acyclics with size less than a given regular cardinal. In effect, the work is to show that the (*set* of) acyclics bounded by some size will generate all of the acyclics. The larger question, of when a set of objects bounded by some size will generate a given localizing subcategory, appears often. The theory of large cardinal numbers is relevant here [BCMR11].

2.3. **Future directions.** The most immediate goal is to continue working to construct aX in the derived category of a ring, or other categories. This will allow me to continue computations of $\mathcal{H}$ and $\mathcal{C}$, and generalize many results from $\mathcal{S}$.

The next hope is that aX will help in understanding localizing subcategories directly. A formal generalization of [HP99, Prop.9.2] states that

$$\mathsf{loc}(X) = \langle aX \rangle \text{ for all } X$$

implies that every localizing subcategory is an HBC. This would be a significant breakthrough.

The next best thing would be progress on certain set theory questions. Is there a set of localizing subcategories? Is there a set of CBCs? A set of CBCs would give a cohomological Bousfield lattice. Toward this end, I am in contact with several set theorists who I met in Barcelona.

3. Metamathematics and contemplative pedagogy

3.1. **Background.** As mentioned earlier, I define metamathematics to be the interdisciplinary study of the mathematical experience. Cognitive scientists [LN01] have looked at how we use conceptual metaphors to understand and communicate advanced mathematical concepts using everyday logic. Raymond Wilder, a mathematician, has done work toward developing a sociological theory of mathematics (see, e.g. [Wil81]). Jacques Hadamard, also a mathematician, analyzed case studies of many well-known mathematicians, to formulate a psychological theory of mathematical invention [Had54]. Several accomplished mathematicians (e.g. [Bye10, Thu94]) have stepped outside of math, to think *about* math. Following their cue, I have attempted to complement my mathematical study and research with metamathematical reading and reflection.

More broadly, I've been tracking the contemplative education movement. In recent decades, a steadily growing base of scientific evidence has shown that contemplative practices (e.g. meditation, yoga, tai chi) have beneficial effects on mental characteristics such as focus, perspective, equanimity, and memory (see [CCMS] for a research bibliography).

The contemplative education movement aims to incorporate appropriate contemplative practices into the classroom. National organizations, such as the Association for Contemplative Mind in Higher Education (ACMHE), are facilitating the development of contemplative pedagogical tools, such as quick pre-lecture meditations, or “one minute papers.” On the ACMHE website, syllabi from a wide range of disciplines are available, including the arts, social sciences, activist programs, and even some science departments. For a fascinating review of the scientific literature related to the use of meditation in higher education, see [SBA08].

3.2. **My contributions.** I believe that a better understanding of *how* we do math will result in better mathematicians, producing better math. I've been trying to convince other mathematicians of this, ever since starting grad school. In 2009, I published a book of essays discussing the relationship between math and music, math and meditation, and math and compassionate work, among other things. The book also included reflection on my own mathematical process, and on the culture of mathematics, and was well-received[1].

In 2008 and 2010 I ran a reading group/seminar based on metamathematical readings, with undergraduates, graduate students, faculty, and staff participating. Since then, I've been compiling a metamathematical reference list, available online [Wol11a].

My most recent project is the Flavors and Seasons blog, an attempt to carefully document the cognitive and meta-cognitive feel of various mathematical experiences. For each distinct experience, I answer eight questions (e.g. What is the logistical context? What thoughts are there? What is the quality of awareness?) The emphasis is on documenting recurring qualities of repeat experiences. The challenge is in self-reflecting and stating the obvious. The hope is to articulate something consistent and universal, rather than purely subjective. Examples of flavors: “working through a proof,” “discussing with a colleague,” “proving myself wrong, via counterexample.” There are currently 26 entries, all available online [Wol11a].

Finally, I am currently pulling together all of these threads, to write a paper on “contemplative mathematics” with Evergreen College professor Brian Walter [WW11]. In it we discuss the role that self-reflection plays and could play in the research community, in the classroom, and in the personal research process.

[1] Reuben Hersh, co-author of *The Mathematical Experience*, which won the National Book Award in 1983 [DH83], told me: “The intention behind *The Mathematical Experience* was to pull aside the veil around the life and work of mathematicians, to show and tell the rest of the world about us. Much of what you write does that, better than we did.”

3.3. **Future directions.** At a recent contemplative education conference, organized by the ACMHE, I began conversations with several science professors about how to incorporate contemplative practice into a math or science classroom. I look forward to exploring these ideas and adapting established contemplative pedagogy to a math context.

The Flavors and Seasons project offers one avenue for this. I am ready to make the project collaborative, and engage colleagues and students with the questions. How do you do math? How does it feel? What is happening in your mind when you're working through a proof, or discussing math with a colleague? It would be fascinating to work with undergraduates on this project, in tandem with a metamathematics seminar.

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