What is Abstraction Theory?

a long answer

Sometimes we're focused on solving the problems in front of us, but sometimes we are free to follow our dreams. In any decision situation - deciding what to cook for dinner, deciding how to make the world a better place, deciding what type of math research to do - there are these two options. You could walk into the kitchen, open the refrigerator, and figure out what leftovers need to be eaten before they go bad. Or you could imagine what you'd like to eat, swing by the store, find a recipe or make it up, and cook an inspired meal. The former is necessary and important, but the latter usually tastes better.

Mathematics is as rich and elaborate as it is because for centuries mathematicians have been working together to build our delicate castles of ideas. The tried-and-true way to contribute to math is to look around, see what questions have been left unanswered or what corner left unexplored, and get to it. This is how all the dark parts in the maps of our knowledge get filled in.

But sometimes mathematicians instead try to build a new castle, or start a new map. The motivation is usually inspired by aesthetics or functionality. Rather than ask, "What can we do?", they ask "What should we do?" "What would I really like math to look like, and how can I make it so?" "It would be really beautiful, or it would be really useful, if math were _____. Can we get there from here?"

The process of abstraction is an example of this latter process. When mathematicians abstract from one system to another, the way they do it is guided by both aesthetics and practicality.

The creation of the negative numbers was an act of abstraction. Our original intuition about numbers comes from piles of rocks and the like. We know how to add one rock to a pile, and we know how to add two piles together. If you start asking questions about removing rocks, soon you need "the collection that is no collection" or "the collection of rocks that, when you add six rocks to it, leaves you with a pile of just one rock". At this point, you leave your intuition behind and take a leap of abstraction, to invent zero and the negative numbers. While they're less useful for counting rocks, the negative numbers complete the positive numbers to give a mathematical object - the integers - that is nicer in some sense. We're so used to thinking about negative numbers, that they don't phase us.

Until the mid-20th century, algebraic topologists had been studying spaces and the properties of spaces. Their spaces all had a good notion of dimension, and those dimensions were always zero or positive. In a huge leap of abstraction, they decided to consider negative-dimensional "space". Why? Partly because it made the math more elegant, in certain ways. But it also allowed for new tools for understanding, and without our tools mathematicians are helpless. When they did this, they had to leave behind some of their intuition about how space works. But this also freed them to explore this new, strange, and beautiful area of math.

Abstraction theory aims to understand the process of abstraction in mathematics. There are many examples, especially in the 20th century, of mathematicians undertaking to cook up a new and fresh model or framework by abstracting from an old one (e.g. the notion of an "abelian category", a "triangulated category", or a "model category").

From the outset, each case seems unique, and mired in history and the whims of humanity. But I believe there are common themes that reveal this as a genuinely mathematical phenomenon. Abstraction theory assumes a dynamic view of mathematical truth, and aims to understand one way that mathematics grows.

The first step in the process seems to be an assessment of what matters - of "what's really going on." This is usually done among the community of specialists, and documented more or less in the mathematical literature. In the topological example, the desire was established to expand from the case ("category") of spaces, but it took a while to figure out the best way to do this. Over several decades, many attempts were put forth in the literature. Eventually, a consensus is more or less reached, and in this case a list of axioms was made. These were sort of a wish-list of properties the new, abstracted category would have, but they functioned just as much as a list of organizational principles for the field - what mattered, what the essence was.

One of the items on the topologists' wish-list was that "suspension becomes invertible." (If you suspend an n-dimensional space X, you get an $(n+1)$-dimensional space ΣX. If suspension was invertible, you could go the other way, and construct an $(n-1)$-dimensional space $\Sigma^{-1}X$. Iterating this would send you into negative dimensions.)

In the process of abstracting our notion of number, ancient mathematicians decided that the operations of addition and subtraction were crucial to the utility of numbers. On their wish-list, they might have asked that "any number added to or subtracted from any other number yields a number." But then, what happens when you subtract five from two?

Once (or if) axioms are established, they seem to allow for new results, and for more elegant proofs of old results. The field becomes more streamlined.

It is another matter to construct a model for these axioms. In other words, what does a "negative-dimensional space" look like? It took algebraic topologists a decade to construct objects that satisfy the axioms they put forth. The result was the stable homotopy category, whose objects are spectra. The ancient mathematicians had to leave behind the idea that numbers are collections of objects, and create the idea that numbers are points on a line.

These examples give a rough outline of some of the ideas that come up in abstraction theory. By studying the many examples of the process of abstraction unfolding in mathematics, the hope is to bring some rigor and self-awareness to the process. The result might be more control over the development of the discipline, and the development of more tractable, more streamlined, and more elegant mathematics.

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