What is Abstraction Theory?

a short answer

Sometimes we're focused on solving the problems in front of us, but sometimes we are free to follow our dreams. In any decision situation - deciding what to cook for dinner, deciding how to make the world a better place, deciding what type of math research to do - there are these two options. You could walk into the kitchen, open the refrigerator, and figure out what leftovers need to be eaten before they go bad. Or you could imagine what you'd like to eat, swing by the store, find a recipe or make it up, and cook an inspired meal. The former is necessary and important, but the latter usually tastes better.

Mathematics is as rich and elaborate as it is because for centuries mathematicians have been working together to build our delicate castles of ideas. The tried-and-true way to contribute to math is to look around, see what questions have been left unanswered or what corner left unexplored, and get to it. This is how all the dark parts in the maps of our knowledge get filled in.

But sometimes mathematicians instead try to build a new castle, or start a new map. The motivation is usually inspired by aesthetics or functionality. Rather than ask, "What can we do?", they ask "What should we do?" "What would I really like math to look like, and how can I make it so?" "It would be really beautiful, or it would be really useful, if math were _____. Can we get there from here?"

The process of abstraction is an example of this latter process. When mathematicians abstract from one system to another, the way they do it is guided by both aesthetics and practicality.

The creation of the negative numbers was an act of abstraction. Our original intuition about numbers comes from piles of rocks and the like. We know how to add one rock to a pile, and we know how to add two piles together. If you start asking questions about removing rocks, soon you need "the collection that is no collection" or "the collection of rocks that, when you add six rocks to it, leaves you with a pile of just one rock". At this point, you leave your intuition behind and take a leap of abstraction, to invent zero and the negative numbers. While they're less useful for counting rocks, the negative numbers complete the positive numbers to give a mathematical object - the integers - that is nicer in some sense. We're so used to thinking about negative numbers, that they don't phase us.

Until the mid-20th century, algebraic topologists had been studying spaces and the properties of spaces. Their spaces all had a good notion of dimension, and those dimensions were always zero or positive. In a huge leap of abstraction, they decided to consider negative-dimensional "space". Why? Partly because it made the math more elegant, in certain ways. But it also allowed for new tools for understanding, and without our tools mathematicians are helpless. When they did this, they had to leave behind some of their intuition about how space works. But this also freed them to explore this new, strange, and beautiful area of math.

Abstraction theory aims to understand the process of abstraction in mathematics. There are many examples, especially in the 20th century, of mathematicians undertaking to cook up a new and fresh model or framework by abstracting from an old one. From the outset, each case seems unique, and mired in history and the whims of humanity. But I believe there are common themes that reveal this as a genuinely mathematical phenomenon. Abstraction theory assumes a dynamic view of mathematical truth, and aims to understand one way in which mathematics grows.

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