What is Abstraction Theory?

a technical answer

I recommend reading the "long answer" to this question first.

At this point, abstraction theory has only been developed to study the process of abstraction in algebra and algebraic topology. It is believed that results found there can be extended to other areas of math, especially other algebraic fields. The main examples used are the development of the notions of: abelian categories, triangulated categories, model categories, and axiomatic stable homotopy categories.

The following is a rough outline of the process of abstraction.

1. One starts with various data, mathematical objects that occur naturally. The prototypical abelian categories were abelian groups and chain complexes; the common ancestor of all axiomatic stable homotopy categories is perhaps the finite CW-complexes.

2. One undertakes some serious soul-searching, to figure out what really matters among the data. What's really going on? What computational tools are indispensable? What structures and properties seem fundamental? What is just so beautiful that it can't be left behind? What is the *essence* of the various data? Furthermore, one indulges in imagination, to ask what should be true? If the data are less than ideal, what would be the ideal?

For abelian categories, kernels, cokernels, monomorphisms, and epimorphisms are some of these essential features. In the category of CW-complexes, it was decided the suspension functor and smash product were key. But algebraic topologists also pined for a category where suspension was invertible and arbitrary coproducts existed. For them, it was time to take a leap of abstraction.

3. In many of these examples, the above conversation centered on coming up with a set of axioms. These axioms might be satisfied by the data from #1, or they might be idealized axioms. In the literature, variations on axiom sets can be found, as mathematicians worked towards answering the questions in #2. In the process of abstraction, the axioms put forth aren't meant to be foundational, in the sense of set theory axioms or geometric axioms; they are more organizational principles about what matters.

Once axioms have been established and more or less agreed upon, there is usually a flood of new proofs. By the nature of their selection, the axioms are able to give more streamlined proofs to many of the fundamental results in that area. For example, in a triangulated category, the functors $[-, X]$ and $[X, -]$ yield long exact sequences, and triangles satisfy the 3×3 lemma. In axiomatic stable homotopy categories, arbitrary products exist, and $\pi_*(-)$ detects isomorphisms. But the axioms also give new results, as in the case of axiomatic stable homotopy categories, where suspension was now invertible.

4. In the case that the axioms are idealized, it is a good idea to find a model for them. There were already categories laying around that satisfied the axioms of an abelian category. But the algebraic topologists had to work for another decade before they could construct a model for their stable homotopy category axioms. There have actually been many different models constructed - e.g. Boardman's spectra, or symmetric spectra, or S-algebras. In a classic text, Adams spends about

40 pages constructing a smash product that did what everyone wanted it to do. Fortunately, the details of the construction are almost irrelevant, since most results only use the smash product axioms.

At this point, you could say that you've abstracted. You've taken the data from #1 and abstracted and generalized. You have a succinct new class of category, with several manifestations, and results that hold universally.

5. The next step is to apply your findings elsewhere. The category of sheaves - obviously quite different from chain complexes - is actually profoundly similar: both are abelian categories. And the derived category of a commutative ring is an axiomatic stable homotopy category! So is the stable comodule category of comodules over a finite-dimensional commutative Hopf algebra (over a field), where we mod out maps that factor through injective comodules.

Although it's common to present mathematical ideas in their most contemporary form, abstraction theory treats ideas as spread across time. Any definition or set of axioms has been arrived at through an involved process of trial and error, and discussion among the community of mathematicians. To flatten the idea to its most recent formulation throws out some of the context, the motivation, and the intuition that drives the idea and gives it power and significance. It's not quite clear to any of us where to draw the line between math and non-math, but it seems like any definition of math ought to include the "out-dated" articles and books that document the evolution of our ideas.

By using these documents to explore the process of math - specifically the process of abstraction - we are attempting to bring rigor and self-awareness to a process that has, until now, seemed ad hoc and historically contingent. This may seem far-fetched, but compared to the state of analysis in the early 1800s, before analysts got serious about rigor, we're not in such bad shape.

If $\#1 - \#5$ are a rough outline of the process of abstraction, then one of the goals of abstraction theory is to apply this process to itself - to abstract the process of abstraction. For step #1, we take as data the development of the notions of abelian, triangulated, model, and axiomatic stable homotopy categories. Step #2 and #3 examine these data and find their essence - the "axioms for the process of abstraction" are basically the items $\#1 - \#5$. For step #4: is it possible to intentionally construct some simple math system that we can then abstract, applying our theory? For step #5: can we apply the theory of abstraction to other areas? Could an abstraction theorist sit down with a group of, say, algebraic geometers, analyze the state of their field, and propose news ways for them to abstract?

Luke Wolcott
December 2010
www.forthelukeofmath.com