What's so interesting about Algebraic Topology?

a long answer

In a straightforward way, I think algebraic topology is interesting because the objects it studies are interesting. It is interesting to investigate our human notion of space. Most of the spaces and objects we study have no correlation to anything in physical reality, and this has kept algebraic topology one of the most out-there and least practical fields of math.

In a less direct way, algebraic topology is interesting because of the way we have chosen to study space. By focusing on the global properties of spaces, the developments and constructions in algebraic topology have been very general and abstract. As an example of this, let me briefly tell you the story of the construction of the category of spectra.

As I explained in the long answer to the question "What is algebraic topology?", algebraic topologists now study spectra - a generalization of spaces to include what can be considered "negative-dimensional space".

In the 1930's, before the invention of spectra, topologists had developed a nice understanding of the global properties of the collection (i.e. "category") of all spaces. For example, this category is almost triangulated, and has arbitrary coproducts. There is an operation between spaces, called smashing, that works nicely, almost like multiplication for spaces. However, there were some imperfections to this category. For example, the category was not quite triangulated - the suspension operation (which is given by smashing with the circle) could be applied to any space, but there was no desuspension. As another example, some cohomology theories on spaces could be represented there, but not all of them.

The meaning of these words is irrelevant to the story. What's relevant is that, presented with this imperfect category, the algebraic topologists decided to create their perfect category.

This, of course, involved a lengthy debate within the community of algebraic topologists, a debate that was essentially aesthetic. What properties would the most beautiful topological category have? They'd been working with spaces, because spaces arise somewhat naturally from our spacial notions derived from the physical world - but what *should* they be working with? What would resonate the most in their minds, and the minds of future mathematicians?

The debate lasted for at least a decade, and eventually a consensus emerged; the axioms for the new category - called the stable homotopy category - were established. One of them was that the category was to be triangulated, and so desuspension was possible - in a sense allowing descent into negative dimensions. Another was that all cohomology theories were to be represented in the category. Mathematicians gleefully set to deriving results and theorems in this new category, many generalizing results from the category of spaces.

It took another decade or so until a model for the axioms was found. In other words, they had made a wish list of properties for a beautiful mathematical universe to have, but was it possible to construct such a universe? What would the objects in that universe be? At this point, we have several constructions, all more or less the same; the objects are called spectra. A spectrum is significantly more complicated than a space, but related. (Roughly, a spectrum is an infinite collection of spaces, of an infinite number of arbitrarily large dimensions, that are related and compounded in a certain way.)

In practice, though, it doesn't really matter that a spectrum is such a complicated object. I can write "Let X and Y be spectra", and ignore the details. I know, for example, that there is an operation, the smash product, and I know from the axioms that it has certain nice properties. So I can say things about the spectrum $X \wedge Y$, that I get from smashing X and Y, without having to deal with all the nitty gritty details of constructing $X \wedge Y$ from X and Y. We are interested in the big picture - the global structure of the collection of all spectra, and because we have created it to have such appealing properties, that structure can be explored without resorting to any ugly computations.

Because of this very holistic approach to understanding mathematics and the properties of mathematical objects, the structures of algebraic topology can be translated into other areas of math. The translation of topological questions into algebraic questions is just one example of this process, which is called category theory. Category theory is a systematic approach to unifying all areas of math within one framework and language, allowing a codification of the deep connections seen between fields of mathematical study. This also appeals to me very much. Algebraic topologists arguably use more category theory than any other type of mathematician, save category theorists.

The question "What is interesting about algebraic topology?", I would argue, is equivalent to the question "What is interesting about algebraic topologists?" It is our personalities, perspectives, and aesthetics that have made the field what it is now. In general, I find algebraic topologists very interesting people. In my experience, they are often big-picture thinkers, trying to get a glimpse of the whole forest. They are often more quirky and idealistic, and less grounded than other mathematicians (which is saying something). Presumably this feedback loop reinforces itself - algebraic topology is abstract and useless, and draws to it sympathetic thinkers who will then make it more so. Finally, in my experience, algebraic topologists tend to be good communicators of mathematics. My theory is that, because of the incredibly abstract nature of our field, there is a need to understand concepts on several different levels of generality, often simultaneously, and to move quickly between these levels. Such a multi-level understanding is also required to communicate mathematics successfully. This has also drawn me to the field.

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March 2010
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