What is Algebraic Topology?

a long answer

Topology is the study of space, and so an algebraic topologist uses algebra to study spaces. For a topologist, a space is just a set of points, with some notion of "closeness" - the definition of closeness is technical, but more or less adheres to the everyday intuitive notion. Mathematicians have thought up all sorts of different spaces, some with quite bizarre properties. For one example, we have a real number line with two distinct zeros infinitely close to each other. Or there's the "long line", which is the same set as the real number line but "longer". Often, our everyday intuitions about how space works fail when we consider the whole range of possible spaces.

Topologists are interested mainly in the global properties of spaces, and this is reflected in our notion of when two spaces are "the same". To a topologist, for example, the surface of a sphere is "the same" as the surface of a football - I can take a sphere and stretch it into a football without tearing. On the other hand, a geometer would say they are different. In fact, different areas of math can be characterized by what objects they study and which of those objects they consider to be "the same".

How do we use algebra to study space? Many properties of spaces can be described using algebraic language. In a very well-defined sense, we take questions about spaces and turn them into algebraic questions. Often, the algebraic questions are easier to answer.

Here's the classic example. To a topologist, a sphere is "the same" as a football (both have no holes), and a donut is "the same" as a coffeecup (both have one hole). A sphere is "different" than a donut, because of that hole. But how does an algebraic topologist make this rigorous? There's an algebraic object, called the "fundamental group", that basically measures how many holes a surface has. And there's a theorem that says that if two surfaces are "the same", then they must have the same fundamental group. We can calculate that the fundamental group of the sphere is zero, and the fundamental group of the donut is $\mathbb{Z} \times \mathbb{Z}$, which is nonzero. Therefore the sphere is not "the same" as the donut.

We took the question "Are the sphere and the donut the same as spaces?" and translated it into "Do the sphere and the donut have the same fundamental group?" This second question was much easier to answer, and because we know some things about that translation process, we could also answer the topological question.

This last part - understanding the translation process between topological objects and questions, and algebraic objects and questions - is very key. The study of this translation process is called category theory, and can be done between virtually any two areas of mathematics. Category theory is responsible for countless beautiful and profound results, connecting vastly different fields of math.

Incidentally, most algebraic topologists these days work not with spaces, but with a sort of generalization of space that can be said to include "negative-dimensional space".

What!? Negative-dimensional space!? I'll try to explain.

As I said, algebraic topologists are mainly interested in global properties of spaces. They often look at the collection of all spaces, and try to say things about the overall structure of that collection. There's a well-understood way of combining spaces, called "smashing", that takes an n-dimensional space and an m-dimensional space, and yields an $(n + m)$-dimensional space. When you smash with a circle, which is one-dimensional, you bump your space up one dimension. Mathematicians started thinking, "Gee, it would be really nice if we could also bump our spaces down one dimension. It'd also be really nice if there was some space that I could smash with, say, a 6-dimensional space, and end up with a 1-dimensional space..." That space, of course, would have to have dimension -5, and spaces as we know them must have non-negative dimension.

But in math you are free to postulate what you want, and so in the 1950s algebraic topologists invented a setting in which the above operations can be done. The objects are no longer spaces, but "spectra". A spectrum is conceptually more complicated than a space, but there's a straightforward way that any space can get turned into a spectrum, and so we think of spectra as generalizations of spaces.

If I had to give a metaphor, I would say that topologists realized they had a sturdy ladder that stretched up through all the non-negative dimensions - going up from one rung to the next was the well-understood operation of smashing with the circle. To get into negative dimensions, they just slid the ladder down, and learned how to climb down the ladder. Of course we can't see and have no intuition about negative dimensions, but with a sturdy ladder we can start to explore them, to think and reason about them.

Amazingly, this process is almost entirely analogous to the invention of negative numbers. Our original intuition about numbers comes from piles of rocks and the like. We know how to add one rock to a pile, and we know how to add two piles together. If you start asking questions about removing rocks, soon you need "the collection that is no collection" or "the collection of rocks that, when you add six rocks to it, leaves you with a pile of just one rock". At this point, you leave your intuition behind and take a leap of abstraction, to invent zero and the negative numbers. While they're less useful for counting rocks, the negative numbers complete the positive numbers to give a mathematical object - the integers - that is nicer in some sense. We're so used to thinking about negative numbers, that they don't phase us. But I believe that if you observe your thoughts very carefully, you'll find that much of your understanding of negative numbers rests on a sort of visual metaphor of the discrete number line, which is nothing other than a ladder, and a descent from the positive along that ladder.

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