What is Algebraic Topology?

a short answer

Topology is the study of space, and so an algebraic topologist uses algebra to study spaces. For a topologist, a space is just a collection of points, with some notion of "closeness" - the definition of closeness is technical, but more or less adheres to the everyday intuitive notion. Mathematicians have thought up all sorts of different spaces, some with quite bizarre properties. Often, our everyday intuitions about how space works fail when we consider the whole range of possible spaces.

How do we use algebra to study space? Many properties of spaces can be described using algebraic language. In a very well-defined sense, we take questions about spaces and turn them into algebraic questions. Often, the algebraic questions are easier to answer.

Here's the classic example. To a topologist, a sphere is "the same" as a football (both have no holes), and a donut is "the same" as a coffeecup (both have one hole). A sphere is "different" than a donut, because of that hole. But how does an algebraic topologist make this rigorous? There's an algebraic object, called the "fundamental group", that basically measures how many holes a surface has. And there's a theorem that says that if two surfaces are "the same", then they must have the same fundamental group. We can calculate that the fundamental group of the sphere is zero, and the fundamental group of the donut is $\mathbb{Z} \times \mathbb{Z}$, which is nonzero. Therefore the sphere is not "the same" as the donut.

We took the question "Are the sphere and the donut the same as spaces?" and translated it into "Do the sphere and the donut have the same fundamental group?" This second question was much easier to answer, and because we know some things about that translation process, we could also answer the topological question.

This last part - understanding the translation process between topological objects and questions, and algebraic objects and questions - is very key. The study of this translation process is called category theory, and can be done between virtually any two areas of mathematics. Category theory is responsible for countless beautiful and profound results, connecting vastly different fields of math.

You might think that translating into algebra seems like a lot of work, just to determine that a donut has a hole and a sphere doesn't. But imagine two 14-dimensional spaces, constructed in very different ways - are they the same or different, and how can you be sure? Translating into algebra may be the only way to know.

Incidentally, most algebraic topologists these days work not with spaces, but with a sort of generalization of space that can be said to include "negative-dimensional space". To read about this, read the long answer to this question.

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