Bousfield lattice invariants of triangulated symmetric monoidal categories

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June 13th, 2015

Special Session: Categorical methods in algebra and topology
AMS-EMS-SPM 2015, Porto

This talk is about work looking at Bousfield lattices in new settings. Some of it appears in [arXiv:1301.4485].

Theorem (Iyengar-Krause 2013).

In any well generated tensor triangulated category there is a set of Bousfield classes, and hence a Bousfield lattice.

PUNCHLINE:

We can now look at Bousfield lattices of new examples.

- quotients and localizations
- proper localizing ideals

back up:

Definition

A tensor triangulated category T is a triangulated category with a closed symmetric monoidal product. We denote this by $\wedge$, and assume that $\wedge$ is compatible with the triangulation, is exact in both variables, and commutes with arbitrary coproducts.

Examples:

- the derived category $D(R)$ of a commutative ring R
- the stable module category $\mathsf{StMod}(kG)$ of a finite group G
- the (p-local) stable homotopy category $\mathfrak{S}$

The Bousfield lattice is a useful invariant of a well generated tensor triangulated category T.

For $X \in \mathsf{T}$, define the *Bousfield class* of X to be

$$\langle X \rangle = \{W \in \mathsf{T} \mid W \wedge X = 0\}.$$

This set has the structure of a complete lattice. Define

$$\langle X \rangle \leq \langle Y \rangle \quad \text{iff} \quad (W \wedge Y = 0 \text{ implies } W \wedge X = 0)\,.$$

Joins are given by coproducts.

$$\bigvee \langle X_i \rangle = \left\langle \coprod X_i \right\rangle$$

Note that $\langle 0 \rangle$ is the minimum class. The meet is defined to be the join of the set of all lower bounds.

This complete lattice is called the *Bousfield lattice* $\mathsf{BL}(\mathsf{T})$ of T.

Given the Iyengar-Krause result, what well generated tensor triangulated categories can we look at?

- quotients and localizations. A Verdier quotient of a well generated category is well generated. Verdier quotients are equivalent to Bousfield localizations.
- a localizing (i.e. closed under triangles and coproducts) subcategory of a well generated category is well generated. Really, we want to look at localizing tensor ideals ($S \subseteq \mathsf{T}$ such that $X \in \mathsf{S}$ and $Y \in \mathsf{T}$ implies $X \wedge Y \in \mathsf{S}$).

First a few comments about quotients and localizations.

Theorem.

Assume T is a well generated tensor triangulated category, and take $\langle W \rangle \in \mathsf{BL}(\mathsf{T})$. Then the quotient map $\pi : \mathsf{T} \to \mathsf{T}/\langle W \rangle$ induces an onto join-morphism that preserves arbitrary joins,

$$\overline{\pi} : \frac{\mathsf{BL}(\mathsf{T})}{(a\langle W \rangle)\downarrow} \to \mathsf{BL}(\mathsf{T}/\langle W \rangle),$$

such that if $\overline{\pi}[\langle X \rangle] = \langle 0 \rangle$ then $[\langle X \rangle] = [\langle 0 \rangle]$.

Here $a\langle W \rangle = \bigvee_{\langle X \wedge W \rangle = \langle 0 \rangle} \langle X \rangle$, and $(a\langle W \rangle)\downarrow = \{\langle Y \rangle : \langle Y \rangle \leq a\langle W \rangle\}$.

Corollary.

If $\langle W\rangle \vee a\langle W\rangle = \langle \mathbb{1}\rangle$, then $\overline{\pi}$ is a lattice isomorphism.

Example.

If $\langle W\rangle = \langle HF_p\rangle$ in $\mathsf{BL}(\mathfrak{S})$, then $\overline{\pi}$ fails to be a lattice isomorphism.

Verdier quotients are equivalent to localized categories. The category $\mathsf{T}/\langle W\rangle$ is equivalent to the image of the Bousfield localization $\mathcal{L} : \mathsf{T} \to \mathsf{T}$ that has acyclics $\langle W\rangle$.

The paper [arXiv:1307.3351] calculated Bousfield lattices of several localized categories of spectra.

Now suppose S is a well generated proper localizing ideal of a well generated tensor triangulated category T. So $\mathbb{1} \notin \mathsf{S}$.

In this case, S is singly generated. That is, $\mathsf{S} = \mathsf{loc}(Z)$ for some $Z \in \mathsf{T}$, the smallest localizing ideal of T containing Z.

Strange stuff happens in $\mathsf{BL}(\mathsf{S})$.

Partial list of strange stuff:

1.

In any BL, the class $\langle 0 \rangle$ is the minimum. With $\mathbb{1} \in \mathsf{T}$, we know

$$\langle X \rangle = \langle 0 \rangle \text{ implies } X = 0,$$

since $0 \wedge \mathbb{1} = 0$ so $W \wedge \mathbb{1} = 0$. This is very useful. With $\mathbb{1} \notin \mathsf{S}$ this can fail in $\mathsf{BL}(\mathsf{S})$.

2.

With $\mathbb{1} \in \mathsf{T}$, the class $\langle \mathbb{1} \rangle$ is the maximum, since $X \wedge \mathbb{1} = 0$ iff $X = 0$. With $\mathbb{1} \notin \mathsf{S}$, we have a different maximum, $\langle \mathsf{Max} \rangle = \vee_{\mathsf{BL}} \langle Y \rangle$. In fact, since $Y \in \mathsf{loc}(Z)$ implies $\langle Y \rangle \leq \langle Z \rangle$, we have $\langle \mathsf{Max} \rangle = \langle Z \rangle$ when $\mathsf{S} = \mathsf{loc}(Z)$.

3.

Care is needed when going between $\mathsf{BL}(\mathsf{S})$ and $\mathsf{BL}(\mathsf{T})$. For $X, Y \in \mathsf{S}$,

$$\langle X \rangle \leq \langle Y \rangle \text{ in } \mathsf{BL}(\mathsf{S}) \text{ does not imply } \langle X \rangle \leq \langle Y \rangle \text{ in } \mathsf{BL}(\mathsf{T}).$$

4.

Complements are strange in $\mathsf{BL}(\mathsf{S})$. First some definitions.

Definitions.

1. Define $\mathsf{DL} = \{\langle X \rangle \in \mathsf{BL} \text{ with } \langle X \rangle = \langle X \wedge X \rangle\}$.
2. A Bousfield class $\langle X \rangle$ is called *complemented* if there exists a class $\langle X^c \rangle$ such that $\langle X \rangle \wedge \langle X^c \rangle = \langle 0 \rangle$ and $\langle X \rangle \vee \langle X^c \rangle = \langle \mathsf{Max} \rangle$. Call $\langle X^c \rangle$ a *complement* of $\langle X \rangle$.
3. Define BA to be the collection of Bousfield classes in DL that are complemented and have a complement in DL.

In the sublattice DL, the meet is given by smashing. Since this is distributive, DL is a frame.

With $\mathbb{1} \notin \mathsf{S}$, a complemented class in $\mathsf{BL}(\mathsf{S})$ may have multiple complements. But if $\langle X \rangle \in \mathsf{BA}$, then its complement in DL is unique, because

$$\langle X^c \rangle = \langle X^c \rangle \curlywedge (\langle X \rangle \vee \langle \tilde{X}^c \rangle) = \langle X^c \rangle \wedge (\langle X \rangle \vee \langle \tilde{X}^c \rangle) = \langle X^c \rangle \wedge \langle \tilde{X}^c \rangle = \langle \tilde{X}^c \rangle.$$

On the other hand, with $\mathbb{1} \in \mathsf{T}$, every complemented class is in DL, because

$$\langle X \rangle = \langle X \wedge \mathbb{1} \rangle = \langle X \rangle \wedge (\langle X \rangle \vee \langle X^c \rangle) = (\langle X \rangle \wedge \langle X \rangle) \vee (\langle X \rangle \wedge \langle X^c \rangle) = \langle X \wedge X \rangle.$$

In this case, all complements are unique, and in fact $\langle X^c \rangle = a \langle X \rangle$.

In either case, BA is a Boolean algebra.

A few quick examples:

1. Let I be the Brown-Comenetz dual of the sphere spectrum in $\mathfrak{S}$. Then $\mathsf{BL}(\mathsf{loc}(I))$ is trivial, because $I \wedge I = 0$.
2. Consider $\mathsf{S} = \mathsf{loc}(HF_p)$ in $\mathfrak{S}$. Then $I \in \mathsf{S}$, and $\langle I \rangle = \langle 0 \rangle$ in $\mathsf{BL}(\mathsf{S})$, although $I \neq 0$.
3. Given a smashing localization $L : \mathsf{T} \to \mathsf{T}$, complete $\mathbb{1} \to L\mathbb{1}$ to a triangle
$$C\mathbb{1} \to \mathbb{1} \to L\mathbb{1}.$$
Then
$$\mathsf{BL}(\mathsf{T}) \cong \mathsf{BL}(\mathsf{loc}(C\mathbb{1})) \times \mathsf{BL}(\mathsf{loc}(L\mathbb{1})) \cong \langle C\mathbb{1} \rangle{\downarrow} \times \langle L\mathbb{1} \rangle{\downarrow}.$$
4. The paper [arXiv:1301.4485] considers BLs of derived categories of several non-noetherian rings.

Thanks for your time.