What are Differential Equations?

a long answer

The study of differential equations is one of the most useful and applicable areas of mathematics. An equation is a mathematical statement that shows a relationship between various different quantities. For example, $E = mc^2$ gives a relationship between energy (E) and mass (m), and says something very powerful about these physical quantities. A *differential equation* relates a quantity to its derivatives.

What's a derivative? Basically, the derivative of a quantity is the instantaneous rate of change of that quantity. For example, velocity is the derivative of position, and acceleration is the derivative of velocity.

Let's build a relatively simple differential equation. Newton's Second Law of Motion say that force is equal to mass times acceleration, or $F = ma$. Imagine a block attached to a spring, hanging from the ceiling. If you pull down on the block (*displace* it), then you feel a pull from the spring. Another law of physics (Hooke's Law) says that this resistive force is proportional to the displacement - the farther down you pull the block, the stronger the string pulls against you. So we have $F = -kx$, where k is some constant, x is displacement, and the negative sign is added since the direction of force is opposite to the direction of displacement (if you pull down the string pulls up; if you push up the spring pushes down).

I told you that acceleration is the derivative of velocity, which is the derivative of position, so we say that acceleration is the second derivative of position, and write $\frac{d^2x}{dt^2} = a$. Putting this together, we get

$$m\frac{d^2x}{dt^2} = -kx.$$

This is a differential equation - an equation relating a quantity (displacement) to its derivatives. The goal of a course in differential equations is to be able to solve such equations. What does it mean to solve a differential equation? Given a differential equation, and some initial conditions (where is the block to start with, and what is its initial velocity?), we can get a formula for the displacement x as a function of time. This formula, in this situation, gives complete knowledge of the position of the block at any given time. Starting with initial conditions and the laws of physics, we arrive at complete knowledge of the behavior of the block. Presto.

In our example, the motion is a nice oscillation of the block up and down. In math language, the solution to this differential equation is sinusoidal.

This example also demonstrates some of the limitations and challenges of differential equations. Each different setup yields a different differential equation, depending on the laws of physics at play. Some differential equations are more difficult to solve than others, and some have even been proven to be unsolvable! Furthermore, in each physical setup we make assumptions that simplify the problem. For example, in real life springs dissipate energy as they move, and as a result the oscillations of the block are dampened. The block will eventually return to its equilibrium position and stop moving. If we take this dampening into effect, the differential equation becomes

$$m\frac{d^2x}{dt^2} = -kx + \gamma\frac{dx}{dt},$$

where γ is some constant. This is a different differential equation that requires a different technique to be solved.

The study of differential equations is a very active area of research - mathematicians are working hard to figure out solutions to all the differential equations that arise in physics, as well as others that haven't arisen yet. An intro class in differential equations works out solutions to the most common and accessible differential equations.

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