# COHOMOLOGICAL BOUSFIELD CLASSES IN STABLE HOMOTOPY CATEGORIES

LUKE WOLCOTT

## 1. Introduction

There is a deep and fruitful analogy between the topological category of spectra and the algebraic derived category of a ring. The category of (p-local) spectra $\mathcal{S}$ is the central object of study in stable homotopy theory, and the (unbounded) derived category of a commutative ring $R$, denoted $D(R)$, has a prominent place in algebra and algebraic geometry. The two categories have significant structural similarities; in the language of [HPS97], both are *monogenic stable homotopy categories*.

It is common to take a somewhat coarse view when investigating the global structure of axiomatic stable homotopy categories; we impose an equivalence relation on the objects, and study the equivalence classes. In fact, there are two types of equivalence relations we can impose - one corresponding to homology, the other to cohomology. The collection of homological Bousfield classes $\mathbf{B}$, those equivalence classes related to homology, have been studied since Bousfield's work in 1979, and have proven to be a useful perspective on understanding global structure. Much is known about $\mathbf{B}$ for the category of p-local spectra $\mathcal{S}$ [Bou79a, Bou79b, HP99], and there are many open conjectures. Dwyer and Palmieri [DP08] recently investigated the (homological) Bousfield classes of the derived category $D(\Lambda)$, where $\Lambda$ is a truncated polynomial algebra on countably many generators.

Every stable homotopy category $\mathcal{C}$ has a ring associated to it, $\pi_*(S^0)$. In the category of spectra, this is the homotopy groups of the sphere spectrum, and in $D(R)$ this is the ring $R$ itself. The category $\mathcal{C}$ behaves differently, depending on whether $\pi_*(S^0)$ is Noetherian or non-Noetherian. Both $\mathcal{S}$ and $D(\Lambda)$ have non-Noetherian $\pi_*(S^0)$, and their Bousfield classes exhibit some similar structure. For example, in both cases the classes fit into a poset that is a complete lattice, called the *Bousfield lattice*. Neeman [Nee92] studied the Bousfield lattice of $D(R)$ when $R$ is Noetherian. In [HPS97, Ch.6], the authors generalized his results to the class of Noetherian stable homotopy categories (for which $\pi_*(S^0)$ is Noetherian) satisfying an additional condition (see Section 5), and are able to completely classify the (homological) Bousfield classes.

Much less is known about the cohomological Bousfield classes. Hovey [Hov95] made some computations in $\mathcal{S}$, and demonstrated some general properties that in fact hold in any stable homotopy category. One interesting result is that, in a stable homotopy category with a proper notion of Brown-Comenetz duality (including $\mathcal{S}$ and $D(R)$ where $R$ is graded connected), all homological Bousfield classes are cohomological Bousfield classes.

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Our goal is to better understand cohomological Bousfield classes in axiomatic stable homotopy categories. In Section 8.2 we give some partial results for a Noetherian stable homotopy category. Our main examples of non-Noetherian stable homotopy categories will be $\mathcal{S}$ and $D(\Lambda)$.

Section 2 gives the definition of a stable homotopy category, and some basic properties. Section 3 defines the two notions of Bousfield equivalence, and explains the type of localizations on stable homotopy categories that motivate these equivalence relations. Section 4 discusses subcategory classification in stable homotopy categories, and the role that Bousfield classes play. The structure of the (homological) Bousfield lattice of a Noetherian stable homotopy category is outlined in Section 5. In Section 6, we describe what is known about the Bousfield lattices of our main non-Noetherian examples $\mathcal{S}$ and $D(\Lambda)$. Section 7 discusses the role Brown-Comenetz duality plays in both these categories. In Section 8, we describe what is known about cohomological Bousfield classes, prove some small results, and ask several questions. Section 9 describes other potential research directions.

## 2. Axiomatic Stable Homotopy Categories

2.1. **Definition.** The categories we are interested in are all examples of monogenic stable homotopy categories. We need some preliminary definitions. Throughout the paper, $[X, Y]$ will denote the set of degree zero morphisms from $X$ to $Y$, and $[X, Y]_*$ the set of all morphisms. Let $\mathfrak{Ab}$ and $\mathfrak{Ab}_*$ denote the categories of abelian groups and graded abelian groups.

*Definition* 2.1. Let $\mathcal{D}$ be a triangulated category. A covariant additive functor $F : \mathcal{D} \to \mathfrak{Ab}$ is called *exact* if for every cofiber sequence

$$X \to Y \to Z \to \Sigma X$$

in $\mathcal{D}$, the following sequence is exact:

$$F(X) \to F(Y) \to F(Z).$$

Similarly, for contravariant functors. An additive functor between two triangulated categories is called exact if it commutes with suspension and sends cofiber sequences to cofiber sequences.

*Definition* 2.2. A *homology functor* is a covariant, exact functor $H : \mathcal{D} \to \mathfrak{Ab}$, such that the canonical map $\coprod H(X_\alpha) \to H(\coprod X_\alpha)$ is equivalence; in other words, $H$ sends coproducts to coproducts.

*Definition* 2.3. A *cohomology functor* is a contravariant, exact functor $H : \mathcal{D} \to \mathfrak{Ab}$, such that the canonical map $H(\coprod X_\alpha) \to \prod H(X_\alpha)$ is equivalence; in other words, $H$ sends coproducts to products.

*Definition* 2.4. We say that a cohomology functor $H : \mathcal{D} \to \mathfrak{Ab}$ is *representable in* $\mathcal{D}$ if there exists an object $Y$ in $\mathcal{D}$ and a natural isomorphism of functors from $H$ to $[-, Y]$. In other words, for every object $X$ in $\mathcal{D}$ we have a functorial isomorphism $H(X) \cong [X, Y]$. In this case, we say that $H$ is *represented by* $Y$.

Brown's Representability Theorem, giving conditions for when a functor on the homotopy category of spaces is representable, is an incredibly powerful tool in algebraic topology. One part of the definition of a stable homotopy category, as we will see, is that all cohomology functors are representable.

*Definition* 2.5. A *closed symmetric monoidal category* is a category $\mathcal{C}$ with

(1) a *sphere object* $S^0$
(2) a functor $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$, denoted $(X,Y)\mapsto X\wedge Y$ and called the *smash product*, that is associative, commutative, and has $S$ as a unit (so $S^0\wedge X\cong X\cong X\wedge S^0$)
(3) for every $Y$ and $Z$ in $\mathcal{C}$, a *function object* $F(Y,Z)$ that is covariantly functorial in $Z$, contravariantly functorial in $Y$, and represents the functor $[-\wedge Y,Z]$. Thus we have a natural isomorphism $[X\wedge Y,Z]\cong[X,F(Y,Z)]$, functorial in each variable.

One consequence of this definition is that the smash product necessarily commutes with arbitrary coproducts. Since we will be working with a triangulated category, we ask that the smash product behaves well with triangles.

*Definition* 2.6. A closed symmetric monoidal structure on a triangulated category is said to be *compatible with the triangulation* if

(1) the smash product commutes with suspension - there is a natural equivalence $\Sigma X\wedge Y\to\Sigma(X\wedge Y)$, and the following diagram commutes:

$$\begin{array}{ccc} S^r\wedge S^s & \xrightarrow{\cong} & S^{r+s} \\ \downarrow c & & \downarrow (-1)^{r+s} \\ S^s\wedge S^r & \xrightarrow[\cong]{} & S^{r+s}, \end{array}$$

where $S^r=\Sigma^r S^0$ and $c$ is the commutativity map.
(2) the smash product is exact, i.e. the functors $X\wedge -$ and $-\wedge X$ are exact for all $X$.
(3) the functors $F(X,-)$ and $F(-,Y)$ are exact (the latter only up to a sign).

In fact, the sphere object is especially nice in a monogenic stable homotopy category. It satisfies the following properties.

*Definition* 2.7. An object $X$ in some category $\mathcal{D}$ is *small* if the natural map $\coprod[X,Y_\alpha]\to[X,\coprod Y_\alpha]$ is an equivalence for all coproducts that exist in $\mathcal{D}$.

*Definition* 2.8. An object $X$ in some category $\mathcal{D}$ is a *graded weak generator* if $[X,Y]_*=0$ implies $Y=0$.

Finally, we're ready to define a monogenic stable homotopy category.

*Definition* 2.9. A *(monogenic) stable homotopy category* is a triangulated category $\mathcal{C}$ with a closed symmetric monoidal structure compatible with the triangulation such that

(1) the sphere object $S^0$ is a small, graded weak generator,
(2) all coproducts of objects of $\mathcal{C}$ exist, and
(3) every cohomology functor on $\mathcal{C}$ is representable.

In [HPS97] a slightly more general definition is given, essentially weakening the requirement that $S^0$ be both small and a weak generator. However, the two cases we're interested in, the category of spectra and the derived category of a ring, are both monogenic. For the rest of this paper, all stable homotopy categories will be assumed to be monogenic. Because $S^0$ is a weak generator, the functor $[S^0, -]_* : \mathcal{C} \to \mathfrak{Ab}_*$ plays a key role in computations, and is denoted $\pi_*(-)$. For each $X$, we call $\pi_*(X)$ the *homotopy groups of* $X$. The homotopy groups of the sphere object, $\pi_*(S^0)$, inherit a ring structure from the map $S^0 \wedge S^0 \to S^0$. (More generally, any "ring object" $R$ in $\mathcal{C}$ will give a ring structure to $\pi_*(R)$.)

**Example 2.10.** Since this definition is motivated by topology, it's no surprise that the category of spectra (*the* stable homotopy category) is a stable homotopy category. So is the category of $p$-local spectra, which we will denote $\mathcal{S}$. These properties of $\mathcal{S}$ were demonstrated clearly in [Ada74]. In particular, he constructs the smash product operation on spectra, and shows that it yields a monoidal structure that is compatible with the triangulation. The sphere object is the sphere spectrum $S^0$, and coproducts are wedges. The ring $\pi_*(S^0)$ is quite complex, to say the least.

**Example 2.11.** It is somewhat surprising that our other main example, the (unbounded) derived category of a commutative ring, satisfies all these conditions. That $D(R)$ is triangulated was well-known; in fact, it was the derived category that motivated Verdier to create the notion of a triangulated category [Ver]. In this case, the smash product is the total tensor product $- \overset{L}{\underset{\Lambda}{\otimes}} -$, with function objects $\mathbb{R}Hom(-,-)$, and it takes some work to see that this is compatible with the triangulation [HPS97, Sect. 9.3]. The sphere object is the ring $R$ itself, or rather the image in $D(R)$ of the chain complex consisting of a single copy of $R$ concentrated in degree zero, and zero modules in every other degree. The ring $\pi_*(S^0)$ is again just $R$. Coproducts are direct sums constructed degree-wise.

In the case of $D(R)$, the homotopy groups are $[S^0, X]_* = [R, X]_* \cong H_*(X)$, just the ordinary homology of $X$ as a chain complex. This shows that $R$ is a weak generator; if $H_*(X) = 0$ then $X$ is exact, and thus equivalent to zero in $D(R)$.

**Example 2.12.** A third example of a monogenic stable homotopy category, which we will only briefly mention, is denoted $\mathcal{C}((kG)^*)$. Here $G$ is a finite $p$-group, $k$ is a field, and $(kG)^*$, the dual of the group algebra, is a commutative Hopf algebra. The objects of $\mathcal{C}((kG)^*)$ are cochain complexes of injective $\mathcal{C}((kG)^*)$-comodules, and morphisms are cochain homotopy classes of maps. In this case, the unit of the smash product is an injective resolution of $k$. Our study of general stable homotopy categories may yield interesting consequences in this particular category.

2.2. **Properties.** Having identified both $\mathcal{S}$ and $D(R)$ as stable homotopy categories, we can use this definition to derive many useful properties of both. The following are proved in [HPS97, Ch. 1]. Let $\mathcal{C}$ be a (monogenic) stable homotopy category.

**Proposition 2.13.** *If a map $f : X \to Y$ induces an isomorphism $\pi_*(f) : \pi_*(X) \to \pi_*(Y)$, then $f$ is an equivalence in $\mathcal{C}$.*

**Proposition 2.14.** *Suppose that $\tau : H \to K$ is a natural transformation of homology or cohomology functors such that $\tau_{S^n} : H(S^n) \to K(S^n)$ is an isomorphism for all $n$. Then $\tau_X$ is an isomorphism for all $X$ in $\mathcal{C}$.*

**Proposition 2.15.** *The category $\mathcal{C}$ is complete, i.e. all products exist.*

**Proposition 2.16.** *The natural map $\coprod X_\alpha \to \prod X_\alpha$ is an equivalence if and only if $\pi_i(X_\alpha) = 0$ for all but finitely many $i$.*

Every object in $\mathcal{C}$ yields both a cohomology theory and a homology theory. Given an object $E$ in $\mathcal{C}$, we define a functor $E^*(X) := [X, E]_*$. This is contravariant and exact, and takes coproducts to products, hence is a cohomology functor. Similarly, the functor $E_*(X) := [S^0, X \wedge E]_*$ is a homology functor. Because all cohomology functors have a representing object, we have a one-to-one correspondence between cohomology functors and objects in $\mathcal{C}$.

The situation with homology theories is more subtle. A *Brown category* is a stable homotopy category in which all homology functors are representable. There are monogenic stable homotopy categories that are not Brown categories. Neeman [Nee95] showed that $D(R)$, for $R = \mathbb{C}[x, y]$, is one such category. However, [Ada74] shows that the category of spectra is a Brown category, and for derived categories we have the following result.

**Proposition 2.17.** [HPS97, Ch.4] *If a commutative ring $R$ is countable, then $D(R)$ is a Brown category.*

This implies that $D(\Lambda)$ is a Brown category (see Section 6.2).

In general, colimits do not necessarily exist in triangulated categories. A *weak colimit* of a diagram is an object with the same universal property as a colimit, except without the uniqueness requirement. If $X$ is a weak colimit of some diagram $\{X_i\}$, and the induced map $\varinjlim \pi_*(X_i) \to \pi_*(X)$ is an isomorphism, then we say $X$ is a *minimal weak colimit.*

**Proposition 2.18.** *In a stable homotopy category, weak colimits exist. Minimal weak colimits exist for sequential diagrams.*

## 3. Bousfield Classes: definition and basic structure.

As noted in the Introduction, it is common to take a somewhat coarser view of stable homotopy categories, by imposing an equivalence relation and working with equivalence classes. There are two different equivalence relations, corresponding to homology and cohomology. Here we introduce both, and discuss the motivation for these definitions and why the homological equivalence classes have been studied more.

### 3.1. (Homological) Bousfield classes.

*Definition* 3.1. Given an object $E$ in a stable homotopy category, define the *(homological) Bousfield class* of $E$ to be the collection

$$\langle E\rangle := \{X \mid E \wedge X = 0\}.$$

Because $S^0$ is a weak generator, $E \wedge X = 0$ if and only if $E_*(X) = \pi_*(E \wedge X) = 0$. Thus $\langle E\rangle$ is the collection of *$E$-acyclics* - the objects that are invisible to the homology functor $E_*$.

For historical reasons, all Bousfield classes will be assumed to be homological, unless specified as cohomological. We say that two objects $E$ and $F$ are *(homologically) Bousfield equivalent* if $\langle E\rangle = \langle F\rangle$.

There is a partial ordering on Bousfield classes, given by reverse inclusion. Thus we say that $\langle E\rangle \leq \langle F\rangle$ when $X \wedge F = 0$ implies $X \wedge E = 0$. The class of the sphere object $\langle S\rangle$ is the maximum class in this ordering, because $X \wedge S = 0$ exactly when $X = 0$, which implies $X \wedge E = 0$ for all $X$. Also, $\langle 0\rangle$ is the minimum.

We can define an operation on Bousfield classes,

$$\langle X\rangle \vee \langle Y\rangle := \langle X \vee Y\rangle,$$

and in a miraculous convergence of notation this is in fact a join operation. Arbitrary joins exist, and are given by $\bigvee_\alpha \langle X_\alpha\rangle = \langle \bigvee_\alpha X_\alpha\rangle$.

In general it is not known whether the collection of Bousfield classes form a set, rather than a proper class. However, Ohkawa [Ohk89] showed that

**Theorem 3.2.** *(Ohkawa) In the category of spectra $\mathcal{S}$, the collection of Bousfield classes is a set.*

In the case of the derived category, we have

**Theorem 3.3.** [DP01] *Every Brown category has a set of Bousfield classes. Thus if $R$ is countable, $D(R)$ has a set of Bousfield classes.*

The non-Noetherian ring $\Lambda$ we are interested in is countable, and so our two main categories of study, $\mathcal{S}$ and $D(\Lambda)$, have sets of Bousfield classes. This allows us to define a meet operation, where $\langle X\rangle \curlywedge \langle Y\rangle$ is the join of (the *set* of) all the lower bounds of $\langle X\rangle$ and $\langle Y\rangle$. Thus in these two examples, the Bousfield classes form a poset, and in fact a lattice called the *Bousfield lattice*.

Another straightforward operation on Bousfield classes is given by $\langle X\rangle \wedge \langle Y\rangle := \langle X \wedge Y\rangle$. This is a lower bound, but in general not the meet.

The structure of the Bousfield lattice in $\mathcal{S}$ was studied extensively in [HP99], and the structure of the Bousfield lattice of $D(\Lambda)$ in [DP08]. These lattices both have rich and subtle structure, and many similarities, to be discussed in Section 6.

### 3.2. Cohomological Bousfield classes.

*Definition* 3.4. Given an object $E$ in a stable homotopy category, define the *cohomological Bousfield class* of $E$ to be the collection

$$\langle E^*\rangle := \{X \mid [X, E]_* = 0\} = \{X \mid E^*(X) = 0\}.$$

Thus $\langle E^* \rangle$ is the collection of $E^*$-acyclics - the objects made invisible by the cohomology functor $E^*$. We say that $E$ and $F$ are *cohomologically Bousfield equivalent* if $\langle E^* \rangle = \langle F^* \rangle$. We give the collection of equivalence classes a partial ordering, again by reverse inclusion. It is not known whether this is a set or a proper class. There are various operations we can put on cohomological Bousfield classes, and these will be discussed in more depth in Section 8, along with some new results.

### 3.3. **Motivation for Bousfield class definition: localization.**

The motivation for the Bousfield equivalence relations comes from the theory of localization.

*Definition* 3.5. Let $H$ be a homology or cohomology functor on a stable homotopy category $\mathcal{C}$.

(1) A map $f : X \to Y$ is an *$H$-equivalence* if the induced map $H(f)$ is an isomorphism.
(2) An object $W$ is *$H$-acyclic* if $H(W) = 0$.
(3) An object $Y$ is *$H$-local* if $[W, Y] = 0$ for all $H$-acyclic $W$.

A *localization functor* $L$ on $\mathcal{C}$ is, briefly, an exact, idempotent functor that inverts a certain class of morphisms, called the $L$-equivalences. Given a localization functor $L$, we call $X$ *$L$-acyclic* if $L(X) = 0$. Localization functors are determined by their acyclics, or by their equivalence morphisms.

**Proposition 3.6.** *Let $L$ and $L'$ be two localization functors. Then $L$ and $L'$ have the same acyclics if and only if they are naturally isomorphic. Also, the set of $L$-equivalences is the same as the set of $L'$-equivalences if and only if $L$ and $L'$ are naturally isomorphic.*

Unfortunately, given a set of morphisms $A$, it is not always possible to construct a localization functor whose equivalences are $A$. And given a collection of objects $B$, it is not always possible to construct a localization functor whose acyclics are $B$.

There is a formal procedure for inverting a set of morphisms, described by Gabriel and Zisman [GZ67], but without some additional conditions this procedure runs into set-theoretical difficulties. The resulting category is not guaranteed to have morphism sets, which is required to stay in this Universe.

However, in [HPS97] the authors show the following.

**Theorem 3.7.** *Consider a homology functor $H$ on a (monogenic) stable homotopy category $\mathcal{C}$, and let $A$ be the set of $H$-equivalences. Then there exists a localization functor $L$ on $\mathcal{C}$ such that $\{L$- equivalences$\} = \{H$- equivalences$\}$.*

We paraphrase this by saying that homology localization functors exist. Bousfield proved this result for the case $\mathcal{C} = \mathcal{S}$ in his pivotal [Bou79b]. Localization at various homology theories has been an extremely powerful tool for understanding $\mathcal{S}$ for several decades, and such localization seems to behave well in an axiomatic stable homotopy category [HPS97, Ch. 3]. It is therefore important and interesting to know when two objects determine the same localization functors. Two objects, $E$ and $F$, determine homology theories $E_*$ and $F_*$, and localization at $E_*$ is the same as localization at $F_*$ precisely when the class of $E$-acyclics is the same as the class of $F$-acyclics, in other words when $\langle E \rangle = \langle F \rangle$.

There is no corresponding theorem for cohomology functors - given a cohomology functor $H$, there is not necessarily a localization functor that inverts precisely the $H$-equivalences. For this reason, the cohomological equivalence relation is less useful and has not been studied as much as the homological one.

Another common use of localization, which we will need later, is localization at a prime ideal of $\pi_*(S^0)$. In order to explain this, we'll need a few definitions.

*Definition* 3.8. In a stable homotopy category, a *ring object* $R$ is an object along with maps $\eta : S^0 \to R$ and $\mu : R \wedge R \to R$ such that the following unit and associativity diagrams commute

$$\begin{array}{ccccc} S^0 \wedge R & \xrightarrow{\eta\wedge 1} & R \wedge R & \xleftarrow{1\wedge\eta} & R \wedge S^0 \\ & \searrow^{\cong} & \downarrow{\scriptstyle \mu} & \swarrow^{\cong} & \\ & & R & & \end{array} \quad \text{and} \quad \begin{array}{ccc} R \wedge R \wedge R & \xrightarrow{1\wedge\mu} & R \wedge R \\ \downarrow{\scriptstyle \mu\wedge 1} & & \downarrow{\scriptstyle \mu} \\ R \wedge R & \xrightarrow{\mu} & R \end{array} .$$

*Definition* 3.9. Let $R$ be a ring object in a stable homotopy category. We say an object $M$ is an *$R$-module object* is there is a map $\nu : R \wedge M \to M$ such that the following unit and associativity diagrams commute

$$\begin{array}{ccc} S^0 \wedge M & \xrightarrow{\eta\wedge 1} & R \wedge M \\ & \searrow^{\cong} & \downarrow{\scriptstyle \nu} \\ & & M \end{array} \quad \text{and} \quad \begin{array}{ccc} R \wedge R \wedge M & \xrightarrow{1\wedge\nu} & R \wedge M \\ \downarrow{\scriptstyle \mu\wedge 1} & & \downarrow{\scriptstyle \nu} \\ R \wedge M & \xrightarrow{\nu} & M \end{array} .$$

If $E$ and $R$ are ring objects, then $E_*(R)$ is a ring in $\mathfrak{Ab}_*$. The ring $\pi_*(R)$ is called the *coefficient ring* of $R$. In particular, the sphere object $S^0$ is always a ring object, and $\pi_*(S^0)$ is a ring. For any object $X$, the map $S^0 \wedge X \xrightarrow{\sim} X$ turns $X$ into a $S^0$-module object, and $\pi_*(X)$ is a module over $\pi_*(S^0)$.

**Proposition 3.10.** *Let L be a localization functor. There is a natural map $LX \wedge LY \to L(X \wedge Y)$. Thus L sends ring objects to ring objects, and module objects to module objects.*

The classical algebraic notion of localization starts with some ring $R$, and some prime ideal $\mathfrak{p} \leq R$, and constructs a new ring $R_\mathfrak{p}$ in which every $r \notin \mathfrak{p}$ is invertible. This localization can be done on $R$-modules, and is exact, in the sense that the $\mathfrak{p}$-localization of a module $M$ is $M_\mathfrak{p} = R_\mathfrak{p} \otimes_R M$ and $R_\mathfrak{p}$ is flat as an $R$-module. Given that $\pi_*(S^0)$ is a ring, and every $\pi_*(X)$ is a module over $\pi_*(S^0)$, it is natural to ask if there is a localization functor on a stable homotopy category that the functor $\pi_*(-)$ will turn into an algebraic localization in $\mathfrak{Ab}$. The answer is yes, and in fact this localization functor is particularly nice.

*Definition* 3.11. A localization functor L is called *smashing* if there is a natural isomorphism $LS \wedge X \to LX$ for all $X$.

**Proposition 3.12.** *Let $\mathfrak{p} \leq \pi_*(S^0)$ be a prime ideal. There is a localization functor $L_\mathfrak{p} : \mathcal{C} \to \mathcal{C}$ such that $\pi_*(L_\mathfrak{p}(X)) = (\pi_*(X))_\mathfrak{p}$, denoted $X \mapsto X_\mathfrak{p}$. This is a smashing localization; i.e. for all $X$ we have $X_\mathfrak{p} = X \wedge S^0_\mathfrak{p}$.*

Localization at a prime ideal of $\pi_*(S^0)$ is an important element in understanding the Bousfield lattice of a Noetherian stable homotopy category, as discussed in Section 5, and one can ask many questions about the effect of localization on Bousfield classes more generally.

## 4. Further Relevance of Bousfield Classes: Subcategory classification

4.1. **Subcategories.** The natural subcollections to study, when considering a stable homotopy category, are those that are closed under the various operations that are possible within such a category. Let $\mathcal{C}$ be a stable homotopy category.

*Definition* 4.1. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *triangulated* if it is closed under the formation of triangles; in other words if $X \to Y \to Z$ is an exact triangle in $\mathcal{C}$ and two of $X$, $Y$, and $Z$ are in $\mathcal{D}$, then so is the third.

*Definition* 4.2. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *thick* if it is triangulated and closed under retracts; i.e. if $X \coprod Y$ is in $\mathcal{D}$, then $X$ and $Y$ are in $\mathcal{D}$.

*Definition* 4.3. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *localizing* if it is thick and closed under the formation of arbitrary coproducts; i.e. $\coprod_\alpha X_\alpha$ is in $\mathcal{D}$ for any collection of $X_\alpha$ in $\mathcal{D}$.

(The Eilenberg swindle [HPS97, Sect. 1.4] shows that any subcategory closed under triangles and coproducts is necessarily closed under retracts.)

Given some collection $A$ of objects in $\mathcal{C}$, the *thick subcategory generated by* $A$, denoted $\mathsf{th}\langle A\rangle$, is the intersection of all the thick subcategories containing $A$. Likewise, we can define the *localizing subcategory generated by* $A$, denoted $\mathsf{loc}\langle A\rangle$. If $X$ and $Y$ are objects in $\mathcal{C}$ and $X$ is in $\mathsf{loc}\langle Y\rangle$, we say that $X$ *can be built from* $Y$.

A classification of such subcategories is very useful in practice, because it is often the case that the properties we are interested in are preserved under the formation of triangles, retracts, or coproducts. For example, consider the property $P$ of having homotopy groups of finite type. Since a cofiber sequence in $\mathcal{C}$ yields a long exact sequence of homotopy groups, we see that property $P$ is preserved under the formation of triangles and retracts. If $X$ in $\mathcal{C}$ happens to have homotopy groups of finite type, then for all $Y$ in $\mathsf{th}\langle X\rangle$, we can conclude that $Y$ has homotopy groups of finite type as well.

Because the smash product is exact and commutes with arbitrary coproducts, every Bousfield class is a localizing subcategory. The study of Bousfield classes offers progress towards the classification of subcategories. For example, in a stable homotopy category in which $R = \pi_*(S^0)$ forms a Noetherian ring (and subject to one more condition - see Section 5), every localizing subcategory is a Bousfield class, and the Bousfield classes form a lattice that is in one-to-one correspondence with subsets of the prime spectrum $\mathsf{Spec\,R}$ of the ring $R$. Furthermore, in this Noetherian setting Neeman [Nee92] showed the correspondence specializes to one between thick subcategories of finite objects in $D(R)$ and subsets of $\mathsf{Spec\,R}$ that are closed under specialization. In fact, Thomason [Tho97] has generalized this to give a classification of thick subcategories of finite objects in $D(R)$ when $R$ is non-Noetherian. These remarkable results will be discussed in more detail in Section 5.

Next we discuss some other examples of subcategory classification.

### 4.2. **Thick subcategories of finite spectra.**

*Definition* 4.4. An object in a stable homotopy category $\mathcal{C}$ is *finite* if it is in $\mathsf{th}\left\langle S^0 \right\rangle$. The collection of finite objects is denoted $\mathcal{F}$.

The finite objects enjoy many nice properties; for example, they are small and *strongly dualizable* (i.e. the natural map $DX \wedge Y \to F(X,Y)$ is an isomorphism for all $Y$, where $DX$ is the Spanier-Whitehead dual $F(X,S^0)$). In fact, in a monogenic stable homotopy category, finite $\Leftrightarrow$ small $\Leftrightarrow$ strongly-dualizable. Since $\mathcal{F}$ is not closed under arbitrary coproducts, the interesting subcategories of $\mathcal{F}$ are the thick subcategories. One of the most significant results, in terms of both elegance and utility, in stable homotopy theory in the last several decades is the classification of the thick subcategories of finite objects in the category of $p$-local spectra [HS98]. (In this paper, all spectra are $p$-local.)

These subcategories are determined by the Morava $K$-theories $K(n)$. For each $n \geq 1$, $K(n)$ is a ring spectrum (actually a *field spectrum* - every module object over $K(n)$ is equivalent to a wedge of suspensions of $K(n)$), and has coefficient ring $\pi_*(K(n)) \cong \mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = 2(p^n - 1)$. The $K(n)$ are constructed from the Brown-Peterson spectrum $BP$. We define $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$, Eilenberg-Maclane spectra. Set $\mathcal{C}_0 = \mathcal{F}$, and for $n \geq 1$ define $\mathcal{C}_n = \{X \text{ in } \mathcal{F} \,:\, K(n-1)_*(X) = 0\} = \langle K(n-1)\rangle \cap \mathcal{F}$.

**Theorem 4.5.** *(Thick Subcategory Theorem)* [HS98] *A subcategory $\mathcal{D}$ of $\mathcal{F}$ is thick if and only if $\mathcal{D} = \mathcal{C}_n$ for some $n$. These subcategories form a nested strictly decreasing filtration of $\mathcal{F}$:*

$$\cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n \subsetneq \mathcal{C}_{n-1} \subsetneq \cdots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0.$$

A spectrum $X$ in $\mathcal{C}_n - \mathcal{C}_{n+1}$ is said to be of type $n$, and we write $\text{type}(X) = n$. Mitchell [Mit85] showed that this filtration is strictly decreasing. Hopkins and Smith use their thick subcategory theorem to prove

**Theorem 4.6.** *(Class-invariance theorem)* [HS98] *Let $X$ and $Y$ be finite spectra. Then $\langle X\rangle \leq \langle Y\rangle$ if and only if type$(X) \geq$ type$(Y)$.*

For each $n \geq 0$, let $F(n)$ denote an arbitrary finite spectrum of type $n$. Thus there is a well-defined class $\langle F(n)\rangle$, and $\langle F(n)\rangle \leq \langle F(m)\rangle$ precisely when $n \geq m$. Every finite spectrum $X$ of type $n$ has $\langle X\rangle = \langle F(n)\rangle$. This gives us a complete understanding of the Bousfield classes of finite spectra.

4.3. **Localizing subcategories when the Bousfield classes form a set.** Every Bousfield class, in any stable homotopy category, forms a localizing subcategory. Because of the various operations and well-understood relations between Bousfield classes, showing the converse - that every localizing subcategory is a Bousfield class - would be a significant step towards a classification of localizing subcategories. This is especially true when the collection of Bousfield classes is a set, because as we mentioned above, in this case the Bousfield classes form a lattice.

**Proposition 4.7.** *Let $\mathcal{C}$ be a stable homotopy category for which the collection of Bousfield classes is a set. Then the following are equivalent.*

1. *Every localizing subcategory of $\mathcal{C}$ is the collection of $E$-acyclics for some $E$ (i.e. is a Bousfield class).*
2. *Every principal localizing subcategory $\mathsf{loc}\langle X\rangle$ is a Bousfield class.*
3. *$\langle X\rangle \leq \langle Y\rangle$ if and only if $X \in \mathsf{loc}\langle Y\rangle$.*

This is proved in [HP99], in the case where $\mathcal{C}$ is the category of spectra, but the proof uses only formal lattice theory results, and therefore carries over to any stable homotopy category with a Bousfield lattice. While a proof of one of these equivalent claims in $\mathcal{S}$ may be too difficult, the derived category of a non-Noetherian ring may be a more tractable setting. Any results towards a classification of localizing subcategories in the derived category of a non-Noetherian ring will likely suggest new approaches to such a classification in $\mathcal{S}$. Evidence for this comes from common structure found in the Bousfield lattices of $\mathcal{S}$ and $D(\Lambda)$, which will be described in Section 6. First, it is worth pointing out what is known about the Bousfield lattice of $D(R)$ for a Noetherian ring $R$.

## 5. The Bousfield Lattice of a Noetherian stable homotopy category

*Definition* 5.1. A stable homotopy category is *Noetherian* if $\pi_*(S^0)$ is a Noetherian ring.

Amnon Neeman, in [Nee92], gives a complete classification of localizing subcategories, and thick subcategories of finite objects, for the derived category $D(R)$ when $R$ is a Noetherian ring. In the derived category, the finite objects are those that are equivalent to a bounded below complex of projectives.

Benson, Carlson, and Rickard, in [BCR97], give a classification of thick subcategories in $\mathcal{C}((kG)^*)$ and their methods bear some similarity to Neeman's. These two categories are both examples of Noetherian stable homotopy categories. In [HPS97], these two examples are generalized, and a classification is given for general Noetherian stable homotopy categories, which we will now describe briefly.

*Definition* 5.2. Let $\mathcal{C}$ be a Noetherian stable homotopy category, with $R = \pi_*(S^0)$. Fix a prime ideal $\mathfrak{p} \leq R$.

1. Write $\mathfrak{p} = (y_1, y_2, ..., y_k)$. Each $y_i$ is a self-map of the sphere. Let $S/y_i$ be the cofiber of the map $S^0 \xrightarrow{y_i} S^0$, and define $S/\mathfrak{p} = S/y_1 \wedge S/y_2 \wedge \cdots S/y_k$. It turns out that different choices of generators $y_i$ yield the same Bousfield class $\langle S/\mathfrak{p}\rangle$, and this is good enough for our purposes.
2. Define $K(\mathfrak{p}) = S^0_{\mathfrak{p}} \wedge S/\mathfrak{p} = (S/\mathfrak{p})_{\mathfrak{p}}$ to be the localization of $S/\mathfrak{p}$ at $\mathfrak{p}$.

**Theorem 5.3.** *The $\langle K(\mathfrak{p})\rangle$ satisfy the following.*

1. *$\langle K(\mathfrak{p})\rangle \wedge \langle K(\mathfrak{p})\rangle = \langle K(\mathfrak{p})\rangle$ for all $\mathfrak{p}$.*
2. *$\langle K(\mathfrak{p})\rangle \wedge \langle K(\mathfrak{q})\rangle = 0$ when $\mathfrak{p} \neq \mathfrak{q}$.*
3. *$\langle S^0\rangle = \coprod_{\mathfrak{p} \in \mathsf{Spec\,R}} \langle K(\mathfrak{p})\rangle$.*

In order to classify the subcategories of $\mathcal{C}$ in a succinct way, we require the following hypothesis: for each $\mathfrak{p} \in \mathsf{Spec}\,\mathsf{R}$, the Bousfield class $\langle K(\mathfrak{p})\rangle$ is minimal among non-trivial Bousfield classes. This hypothesis is satisfied by both $\mathcal{C}((kG)^*)$ and $D(R)$, for Noetherian $R$.

**Theorem 5.4.** *Suppose that each $\langle K(\mathfrak{p})\rangle$ is minimal. Then every localizing subcategory is a Bousfield class, and the Bousfield classes form a lattice. The Bousfield lattice is in one-to-one correspondence with the subsets of $\mathsf{Spec}\,\mathsf{R}$. The lattice of thick subcategories of finite objects is in one-to-one correspondence with the subsets of $\mathsf{Spec}\,\mathsf{R}$ that are closed under specialization.*

Recall that a subset $T \subseteq \mathsf{Spec}\,\mathsf{R}$ is *closed under specialization* if $\mathfrak{p} \in T$ and $\mathfrak{p} \leq \mathfrak{q}$ implies that $\mathfrak{q} \in T$. This is equivalent to $T$ being a union of Zariski closed sets.

The bijection is given in terms of supports.

*Definition* 5.5. Given an object $X$ in $\mathcal{C}$, the *support of* $X$ is

$$\mathrm{supp}(X) = \{\mathfrak{p} \mid K(\mathfrak{p})_*(X) \neq 0\}.$$

If $\mathcal{D}$ is a subcategory of $\mathcal{C}$, define $\mathrm{supp}(\mathcal{D}) = \bigcup_{X\in\mathcal{D}} \mathrm{supp}(X)$.

Let $\mathcal{A}$ be a localizing subcategory of $\mathcal{C}$, and let $T$ be a subset of $\mathsf{Spec}\,\mathsf{R}$. The first correspondence in the theorem is given by the following:

$$\{\text{localizing subcategories of } \mathcal{C}\} \longleftrightarrow \{\text{subsets of } \mathsf{Spec}\,\mathsf{R}\},$$

$$\mathcal{A} \mapsto \mathrm{supp}(\mathcal{A}) = \{\mathfrak{p} \mid S/\mathfrak{p} \in \mathcal{A}\} \subseteq \mathsf{Spec}\,\mathsf{R},$$

and

$$\mathsf{loc}\,\langle K(\mathfrak{p}) \mid \mathfrak{p} \in T\rangle = \langle W\rangle \leftarrow T,$$

where $W = \coprod_{\mathfrak{q}\notin T} K(\mathfrak{q})$.

When we restrict to finite objects, the correspondence becomes

$$\{\text{thick subcategories of } \mathcal{F}\} \longleftrightarrow \{\text{subsets of } \mathsf{Spec}\,\mathsf{R} \text{ closed under specialization}\},$$

$$\mathcal{A} \mapsto \mathrm{supp}(\mathcal{A}) = \{\mathfrak{p} \mid S/\mathfrak{p} \in \mathcal{A}\} \subseteq \mathsf{Spec}\,\mathsf{R},$$

and

$$\mathsf{th}\,\langle S/\mathfrak{p} \mid \mathfrak{p} \in T\rangle = \{W \text{ in } \mathcal{F} \mid \ \mathrm{supp}(W) \subseteq T\} \leftarrow T.$$

Both these correspondences are order-preserving bijections of posets. Although tangential to our investigation of non-Noetherian stable homotopy categories, [HPS97] give several other strong results for Noetherian stable homotopy categories, such as

**Theorem 5.6.** *Let $\mathcal{C}$ be a Noetherian stable homotopy category in which each $\langle K(\mathfrak{p})\rangle$ is minimal. The telescope conjecture holds (i.e., every smashing localization is a finite localization). Also, the objects $K(\mathfrak{p})$ detect nilpotence.*

5.1. **Generalization to Non-noetherian $D(R)$.** In general, the techniques used to prove the above strong results for Noetherian stable homotopy categories will not carry over to the non-Noetherian case. However, there is one example of a result that generalizes nicely. Thomason [Tho97] gave a classification for thick subcategories of finite objects in $D(R)$ for an arbitrary commutative ring $R$, which reduces to the above result in the case where $R$ is Noetherian.

**Theorem 5.7.** [Tho97] *Let $R$ be any commutative ring. There is a one-to-one correspondence between thick subcategories of finite objects in $D(R)$ and subsets of $\mathsf{Spec\,R}$ of the form $\bigcup_\alpha V(I_\alpha)$, where each $I_\alpha$ is finitely generated.*

Just as above, a thick subcategory $\mathcal{A}$ of $\mathcal{F}$ corresponds to $\mathrm{supp}(\mathcal{A}) \subseteq \mathsf{Spec\,R}$, and a subset $T \subseteq \mathsf{Spec\,R}$ corresponds to $\{W \text{ in } \mathcal{F} \mid \ \mathrm{supp}(W) \subseteq T\}$.

Note that in the case where $R$ is Noetherian, every subset of the form $\bigcup_\alpha V(I_\alpha)$ is closed under specialization, because ideals in a Noetherian ring are finitely generated.

## 6. The Bousfield Lattice when $\pi_*(S^0)$ is non-Noetherian

In this section, we'll outline what is known about the Bousfield lattice for our two main non-Noetherian examples, $\mathcal{S}$ and $D(\Lambda)$.

6.1. **Spectra.** Bousfield introduced the notion of Bousfield classes on $\mathcal{S}$ in [Bou79a] and [Bou79b], and established many results. Much more was made possible by Ohkawa's proof that there is only a set of Bousfield classes of spectra [Ohk89]. This gives the collection of Bousfield classes, denoted **B**, the structure of a *complete lattice* - a poset with finite meets and arbitrary joins.

As mentioned in Section 3, the smash product induces an operation on **B**, given by $\langle X\rangle \wedge \langle Y\rangle := \langle X \wedge Y\rangle$. While this is a lower bound, it is in general not the meet. The smash product distributes over arbitrary joins, but in general the meet operation does not. However, there is a nice sub-poset within **B** in which it does. Let **DL** be the collection of Bousfield classes $\langle E\rangle$ such that $\langle E\rangle = \langle E\rangle \wedge \langle E\rangle$.

**Proposition 6.1.** *In **DL**, the meet of $\langle X\rangle$ and $\langle Y\rangle$ is $\langle X\rangle \wedge \langle Y\rangle$. Thus **DL** is a frame, i.e. a complete lattice in which the meet distributes over arbitrary joins. The inclusion $i : \boldsymbol{DL} \hookrightarrow \boldsymbol{B}$ preserves arbitrary joins but does not preserve meets.*

Bousfield shows that every ring spectrum and every finite spectrum is in **DL**. The Brown-Comenetz dual $I$ of the sphere (see Definition 6.7) is not in **DL**, since $I \wedge I = 0$.

*Definition* 6.2. A subset $J$ of a complete lattice is a *complete ideal* if it is closed under arbitrary joins, and if $x \in J$ and $y \leq x$ implies $y \in J$.

*Definition* 6.3. A spectrum $E$ is *strange* if $\langle E\rangle < \langle H\mathbb{F}_p\rangle$.

Let $J$ be the smallest complete ideal of $\mathbf{B}$ containing the classes of all the strange spectra.

There is a retract $r$ of $\mathbf{B}$ onto $\mathbf{DL}$, the right adjoint to inclusion. Hovey and Palmieri [HP99] show this induces an epimorphism $r' : \mathbf{B}/J \to \mathbf{DL}$, and conjecture that $r'$ is an isomorphism of lattices. One interesting consequence of this conjecture is that for all $E$ and $n \geq 2$, we would have $\langle E \rangle^{\wedge n} = \langle E \rangle^{\wedge (n+1)}$.

Because the smash product distributes over arbitrary joins, we can define a useful complementation operator $a$: let $a \langle X \rangle$ be the join of all $\langle Y \rangle$ such that $\langle X \rangle \wedge \langle Y \rangle = \langle 0 \rangle$. This operation has the following properties.

**Lemma 6.4.**
1. *$\langle E \rangle \leq a \langle X \rangle$ if and only if $E \wedge X = 0$.*
2. *$a$ is order-reversing: $\langle X \rangle \leq \langle Y \rangle$ if and only if $a \langle X \rangle \geq a \langle Y \rangle$.*
3. *$a^2 \langle X \rangle = \langle X \rangle$.*
4. *$\langle X \rangle \curlywedge \langle Y \rangle = a (a \langle X \rangle \vee a \langle Y \rangle)$.*
5. *$a$ converts arbitrary joins to meets and arbitrary meets to joins.*

*Definition* 6.5. A Bousfield class $\langle X \rangle$ is *complemented* if there exists a class $\langle Y \rangle$ such that $\langle X \rangle \vee \langle Y \rangle = \langle S^0 \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle 0 \rangle$.

Using the lemma, it is not hard to show that if $\langle X \rangle$ is complemented, then its complement is $a \langle X \rangle$. Let $\mathbf{BA}$ denote the collection of all complemented Bousfield classes. This sub-poset of $\mathbf{B}$ is particularly nice.

**Lemma 6.6.** *Suppose that $\langle X \rangle$ and $\langle Y \rangle$ are in $\boldsymbol{BA}$, and $\langle E \rangle$ is an arbitrary Bousfield class.*
1. *$\langle E \rangle \leq \langle X \rangle$ if and only if $\langle E \rangle = \langle E \rangle \wedge \langle X \rangle$.*
2. *$\langle X \rangle \curlywedge \langle Y \rangle = \langle X \rangle \wedge \langle Y \rangle$.*
3. *$\boldsymbol{BA} \subseteq \boldsymbol{DL}$.*
4. *$\boldsymbol{BA}$ is a Boolean algebra; i.e. a distributive lattice in which every element has a complement.*

We know that every finite spectrum is in $\mathbf{BA}$, but the inclusion $\mathbf{BA} \subset \mathbf{DL}$ is proper since for example $\langle H\mathbb{Z} \rangle$ is a ring spectrum not in $\mathbf{BA}$.

In studying $\mathbf{B}$, Brown-Comenetz duality arises as a useful tool.

*Definition* 6.7. Given a spectrum $X$, the *Brown-Comenetz dual $IX$* is the representing object for the cohomology functor

$$Y \mapsto \operatorname{Hom}\left(\pi_0(X \wedge Y), \mathbb{Q}/\mathbb{Z}_{(p)}\right),$$

where $\mathbb{Z}_{(p)}$ is the $p$-localization of $\mathbb{Z}$.

Let $I := IS^0$. It's not hard to show that in general $IX = F(X, I)$. Furthermore, $IX = 0$ if and only if $X = 0$. We will discuss Brown-Comenetz duality further in Section 7. Here we only mention a few properties of $I$.

**Lemma 6.8.** *Let $I$ be the Brown-Comenetz dual of the sphere $S^0$.*
1. *$I$ is strange.*
2. *$I \wedge I = 0$, $H\mathbb{F}_p \wedge I = 0$, and $BP \wedge I = 0$.*
3. *$\langle IF \rangle = \langle I \rangle$ for all finite spectra $F$.*
4. *Every finite-dimensional torsion spectrum is $I$-local.*
5. *$\langle E \rangle \geq \langle I \rangle$ if and only if $E$ has a finite local, i.e. there is a non-zero finite spectrum $X$ that is $E$-local.*

One more important property of $I$ hinges on the following conjecture of Hovey.

**Conjecture 6.9.** (The Dichotomy Conjecture). Every spectrum has either a finite local or a finite acyclic.

**Lemma 6.10.** *If the Dichotomy Conjecture holds, then $I$ is a minimal Bousfield class.*

We believe that Brown-Comenetz duality offers a useful tool for understanding the structure of the Bousfield lattice, and that many conjectures about that structure can be framed in terms of Brown-Comenetz duality. Further evidence for this approach come from studying the Bousfield lattice of the stable homotopy category $D(\Lambda)$.

6.2. **Truncated polynomial algebra on countably many generators.** Fix a countable field $k$ and a sequence of integers $n_i \geq 2$ for $i \geq 1$. Define the $k$-algebra $\Lambda$ by

$$\Lambda = k[x_1, x_2, x_3, ...]/\,(x_i^{n_i} \text{ for all } i)\,,$$

with $\deg x_i = 2^i$. The (unbounded) derived category $D(\Lambda)$ is a (monogenic) stable homotopy category, with sphere object $S^0 = \Lambda$ and $\pi_*(S^0) = [\Lambda, \Lambda]_* = \Lambda$. Dwyer and Palmieri [DP08] have investigated the Bousfield lattice of $D(\Lambda)$; in this section we outline some of their results and methods.

The motivation for choosing this ring is that it is non-Noetherian, locally finite, graded connected, graded commutative, and has one prime ideal (generated by the elements of positive degree). The same is essentially true of the homotopy groups of the $p$-sphere spectrum $\pi_*(S^0)$ in $\mathcal{S}$.

Let $\mathbf{B}$ be the collection of Bousfield classes of $D(\Lambda)$. Because $\Lambda$ is countable, $\mathbf{B}$ is a set and not a proper class. As with $\mathcal{S}$, this implies that $\mathbf{B}$ is a complete lattice. One of the interesting results in [DP08] is

**Theorem 6.11.** *The Bousfield lattice $\boldsymbol{B}$ of $D(\Lambda)$ has cardinality $2^{2^{\aleph_0}}$.*

This shows that the Bousfield lattice of $D(\Lambda)$ is quite different than that of a Noetherian ring. With a Noetherian ring, the Bousfield lattice is limited by $\mathsf{Spec}\,\mathsf{R}$. For example, consider the Noetherian rings

$$\Lambda_m = k[x_1, x_2, ..., x_m]/\,(x_i^{n_i} \text{ for all } i)\,.$$

From Section 5 we know that the Bousfield lattice of each $D(\Lambda_m)$ has only two classes: $\langle 0 \rangle$ and $\langle \Lambda_m \rangle$.

Results and proof techniques used in the category $D(\Lambda)$ may be applicable to $\mathcal{S}$. For example, the following interesting result in $D(\Lambda)$ might possibly be translatable into a statement about spectra.

**Theorem 6.12.** *In $D(\Lambda)$, there are objects of arbitrarily high smash-nilpotence height. That is, for any $n \geq 1$ there is an object $X_n$ in $D(\Lambda)$ such that the $n$-fold smash product of $X_n$ with itself is nonzero, while the $(n+1)$-fold smash product is zero.*

Most of the proofs in [DP08] use the following constructions. Let $S$ be any subset of the natural numbers $\mathbb{N}$. Define objects in $D(\Lambda)$

$$\Lambda(S) = k[x_i \,:\, i \in S]/\,(x_i^{n_i})\,.$$

For any object $X$ in $D(\Lambda)$, define the Brown-Comenetz dual $IX$ to be the representing object for the cohomology functor

$$Y \mapsto \operatorname{Hom}_k(\pi_*(X \wedge Y), k) = \operatorname{Hom}_k\left(H^*(X \otimes^L_\Lambda Y), k\right).$$

An application of tensor-hom adjointness shows that this gives $IY = \mathbb{R}Hom_k(Y, k)$. Note that this $\mathbb{R}Hom$ is taken in $D(k)$ and considered as an object of $D(\Lambda)$ via the inclusion induced by the degree zero map $k \hookrightarrow \Lambda$. This is because we've defined Brown-Comenetz duality using $k$-vector space maps, as opposed to $\Lambda$-module maps. This choice will be explained in Section 7.

For each subset $S \subseteq \mathbb{N}$, let $I(S) = I(\Lambda(S)) = \mathbb{R}Hom_k(\Lambda(S), k) = \operatorname{Hom}_k(\Lambda(S), k)$, the graded $k$-dual of $\Lambda(S)$ (concentrated in non-positive degrees).

The objects $\Lambda(S)$ and $I(S)$, determined by subsets of $\mathbb{N}$, are useful tools for understanding the Bousfield lattice of $D(\Lambda)$.

*Definition* 6.13. Given subsets $S, T \subseteq \mathbb{N}$, we say that $T$ is *cofinite* in $S$ if $T \subseteq S$ and the complement of $T$ in $S$ is finite. We say that $S$ and $T$ are *commensurable*, written $S \sim T$, if $S \cap T$ is cofinite in both $S$ and $T$. We write $S \lesssim T$ if $S$ is commensurable with a subset of $T$.

**Proposition 6.14.** *Fix subsets $S$ and $T$ of $\mathbb{N}$.*

(1) *$\Lambda(S)$ is in the thick subcategory generated by $\Lambda(T)$ if and only if $T$ is cofinite in $S$.*
(2) *The following are equivalent:*
  (a) *$S \lesssim T$.*
  (b) *$\Lambda(S)$ can be built from $\Lambda(T)$.*
  (c) *$\langle \Lambda(S) \rangle \leq \langle \Lambda(T) \rangle$.*

**Proposition 6.15.** *Fix subsets $S$ and $T$ of $\mathbb{N}$.*

(1) *$I(S)$ is in the thick subcategory generated by $I(T)$ if and only if $T$ is cofinite in $S$.*
(2) *The following are equivalent:*
  (a) *$T \lesssim S$.*
  (b) *$I(S)$ can be built from $I(T)$.*
  (c) *$\langle I(S) \rangle \leq \langle I(T) \rangle$.*

**Proposition 6.16.** *Fix subsets $S$ and $T$ of $\mathbb{N}$.*

(1) *$\Lambda(S) \otimes^L_\Lambda I(T) = 0$ if and only if $S^c \cap T$ is infinite.*
(2) *$I(S) \otimes^L_\Lambda I(T) = 0$ if and only if $S \cup T$ is infinite.*
(3) *$\Lambda(S) \otimes^L_\Lambda \Lambda(T) \neq 0$ for all $S$ and $T$.*

*Definition* 6.17. For any subset $S \subseteq \mathbb{N}$, let $k(S)$ denote the trivial $\Lambda(S)$-module $k$. Given a partition $\mathbb{N} = A \coprod B \coprod C$, define a module $M_{A,B,C}$ over $\Lambda \cong \Lambda(A) \otimes \Lambda(B) \otimes \Lambda(C)$ by

$$M_{A,B,C} := \Lambda(A) \otimes k(B) \otimes I(C).$$

Here all tensor products are over $k$. We can use the last two propositions to get a similar proposition regarding the $M_{A,B,C}$.

**Proposition 6.18.** *Given partitions $A \coprod B \coprod C$ and $A' \coprod B' \coprod C'$ of $\mathbb{N}$, the following are equivalent.*

(1) *$(A, B, C)$ is less than or equal to $(A', B', C')$ in the left lexicographic commensurability order - i.e. either $A \lesssim A'$ and $A \nsim A'$, or $A \sim A'$ and $B \lesssim B'$.*

(2) $M_{A,B,C}$ *may be built from* $M_{A',B',C'}$.
(3) $\langle M_{A,B,C}\rangle \leq \langle M_{A',B',C'}\rangle$.

Thus the $\Lambda$-modules $M_{A,B,C}$ comprise a well-understood sub-lattice $\mathbf{M}$ within $\mathbf{B}$. Is it possible that in fact $\mathbf{M} = \mathbf{B}$? In other words, is every element of $D(\Lambda)$ Bousfield equivalent to a direct sum of $M_{A,B,C}$'s? An interesting question hovering in the background is

**Question 6.19.** Let $R$ be a commutative ring. Is every object in $D(R)$ Bousfield equivalent to an $R$-module?

For the category $D(\Lambda)$, the Brown-Comenetz dual of the sphere object is $I(\Lambda) = I(\Lambda(\mathbb{N}) = I(\mathbb{N})$. In analogy with $\mathcal{S}$ and the Dichotomy Conjecture, one wonders if $I(\mathbb{N})$ is a minimal class. In fact, Dwyer and Palmieri show that

**Proposition 6.20.** *$I(\mathbb{N})$ is the minimum nonzero Bousfield class; for every nonzero object $E$, we have $\langle E\rangle \geq \langle I(\mathbb{N})\rangle$.*

And as a corollary this implies that

**Corollary 6.21.** ***BA*** *is trivial in the Bousfield lattice for $D(\Lambda)$; the only complemented classes are $\langle 0\rangle$ and $\langle\Lambda\rangle$.*

## 7. Brown-Comenetz duality

Brown-Comenetz duality shows up in both these examples of non-Noetherian stable homotopy categories, as a way to construct interesting objects and as a way of stating structural results. This suggests looking at Brown-Comenetz duality in an abstract stable homotopy category, and trying to understand its role in the Bousfield lattice.

The two definitions for Brown-Comenetz duality in $\mathcal{S}$ and $D(\Lambda)$ are different. What properties of this functor $I(-)$ are used in proofs? For one, $I(-)$ must be exact and contravariant. It must be the case that $IX = 0$ if and only if $X = 0$. In computations, we often use $IX = F(X, I)$, where $I = I(S^0)$. There is a natural map $X \to I^2X$, which is an isomorphism when $X$ satisfies certain finiteness conditions. These seem to be the main properties of $I(-)$ that are used.

In the category of spectra we have

$$[X, I(Y)] \cong \operatorname{Hom}\left(\pi_0(X\wedge Y), \mathbb{Q}/\mathbb{Z}_{(p)}\right).$$

This is exact, because $\mathbb{Q}/\mathbb{Z}_{(p)}$ is an injective $\mathbb{Z}_{(p)}$-module. It satisfies $IX = 0 \Leftrightarrow X = 0$, because $\mathbb{Q}/\mathbb{Z}_{(p)}$ is a cogenerator in the category of $p$-local abelian groups, and the $\pi_0(X\wedge Y)$ term is a natural way to get an exact covariant functor of both $X$ and $Y$ that lands in this category. Furthermore, this $X\wedge Y$ term is the source of the relationship $IX = F(X, I)$, where $I = I(S^0)$. Note that $\mathbb{Z}_{(p)}$ is $\pi_0(S^0)$ in the (p-local, as always) category of spectra. The map $X \to I^2X$ is an isomorphism when the homotopy groups of $X$ are finite.

The definition in $D(\Lambda)$ has

$$[X, I(Y)] \cong \operatorname{Hom}_k\left(\pi_*(X\wedge Y), k\right) = \operatorname{Hom}_k\left(H^*(X\otimes^L_\Lambda Y), k\right).$$

This seems to be a good choice for the definition, since it gives the useful form $IX = \mathbb{R}Hom_k(X,k)$. Of course, $k$ is an injective cogenerator in the category of $k$-modules. Note that $k$ is $\pi_0(S^0)$ in $D(\Lambda)$. The map $X \to I^2X$ is an isomorphism when $X$ is finite-dimensional in each degree.

The rule seems to be that in a stable homotopy category $\mathcal{C}$ for which $\pi_*(S^0)$ is graded connected, we should look at the degree zero piece $\pi_0(S^0)$, find an injective cogenerator $K$ in the category of $\pi_0(S^0)$-modules, and make $I(Y)$ to represent

$$X \mapsto \operatorname{Hom}_{\pi_0(S^0)}(\pi_0(X \wedge Y), K).$$

In fact, we can construct a functor $I(-)$ that satisfies the properties mentioned above in a more general stable homotopy category $\mathcal{C}$ as follows. Choose $I$ to be a cogenerator in $\mathcal{C}$, in the sense that $F(X,I) = 0$ implies $X = 0$. Then define $I(X) = F(X,I)$. This gives an exact contravariant functor such that $IX = 0 \Leftrightarrow X = 0$. We also have a natural map $X \to I^2X$, corresponding to the identity $IX \to IX$ in

$$[X, I^2X] = [X, I(IX)] = [X, F(IX, I)] = [X \wedge IX, I] = [IX \wedge X, I] = [IX, IX].$$

However, we don't know when this map is going to be an isomorphism.

Using Brown-Comenetz duality, it seems that one can say interesting things about the relationship between homological and cohomological Bousfield classes. For example, we have

**Proposition 7.1.** *Let $\mathcal{C}$ be a stable homotopy category with a Brown-Comenetz functor $I(-)$ as defined above. Every homological Bousfield class is a cohomological Bousfield class. Indeed, $\langle E\rangle = \langle (IE)^*\rangle$.*

**Proof.** We have $W \wedge E = 0$ if and only if $0 = I(W \wedge E)$, and $I(W \wedge E) = F(W \wedge E, I) = F(W, F(E,I)) = F(W, IE)$. Since $\mathcal{C}$ is assumed to be monogenic, $F(W, IE) = 0$ if and only if $[W, IE] = 0$. □

Given the role that $IS^0$ plays in an understanding of the Bousfield lattice of $\mathcal{S}$, and that the objects $I(\Lambda(S))$ play in an understanding of the Bousfield lattice of $D(\Lambda)$, we would like to understand the way that a Brown-Comenetz duality functor $I(-)$ operates on the collection of (homological) Bousfield classes, or the collection of cohomological Bousfield classes.

## 8. Cohomological Bousfield classes

Here we describe what is known about cohomological Bousfield classes, including some new results in a Noetherian stable homotopy category, and ask several questions about the relationship between homological Bousfield classes and cohomological Bousfield classes.

As mentioned in Section 3.2, it's not known in general when the collection of cohomological Bousfield classes forms a set rather than a proper class. However, it's not hard to see that every cohomological Bousfield class is a localizing subcategory, closed under triangles and coproducts. If $\mathcal{C}$ is a Noetherian stable homotopy category, in which the $\langle K(\mathfrak{p})\rangle$ are minimal classes, then we have a classification of localizing subcategories via the prime spectrum of $\pi_*(S^0)$, and hence the collection of cohomological Bousfield classes is necessarily a set.

Here are some other properties, from [Hov95], that apply in a general stable homotopy category.

**Lemma 8.1.** *Let $\mathcal{C}$ be a stable homotopy category.*

1. *Any object $E$ is $E^*$-local.*
2. *If $X \to Y \to Z$ is a cofiber sequence, then $\langle Y^* \rangle \leq \langle X^* \rangle \vee \langle Z^* \rangle$.*
3. *If $Y$ is a retract of $X$, then $\langle Y^* \rangle \leq \langle X^* \rangle$.*
4. *$X$ is $E^*$-local if and only if $\langle X^* \rangle \leq \langle E^* \rangle$.*

Besides this, little is known about cohomological Bousfield classes. There may be several reasons for this. For one, there is no evidence that cohomological localization functors exist. However, recently Casacuberta, Scevenels, and Smith [CSS05] have done some interesting work towards answering this question in $\mathcal{S}$, showing that the answer hinges on accepting a large-cardinal axiom (Vopěnka's principle) that is by all indications independent of the ZFC axioms.

Another reason that cohomological Bousfield classes have been neglected is that they don't appear to behave as well as their homological cousins. For example, there doesn't seem to be any straightforward way of using the smash product to define an operation $\langle X^* \rangle \wedge \langle Y^* \rangle$. Hovey shows in [Hov95] that $\langle BP^* \rangle$ is incomparable with $\langle K(n)^* \rangle$, despite the fact that $K(n)$ is a module over $BP$; in the case of homological Bousfield classes $\langle R \rangle \geq \langle M \rangle$ for every ring object $R$ and $R$-module object $M$. Also, $\langle BP^* \rangle$ is incomparable with $\langle (BP \wedge K(n))^* \rangle$, whereas of course $\langle X \rangle \geq \langle X \wedge Y \rangle$ for all $X$ and $Y$.

It is our belief that there is more to be said about cohomological Bousfield classes. Just as one can find the frame **DL** and the Boolean algebra **BA** within the Bousfield lattice of spectra, it seems that with some restriction on the objects one considers, it is possible to recover nice structure within the collection of cohomological Bousfield classes.

For example, the collection of objects $X$ for which the map $X \to I^2X$ is an isomorphism may yield particularly nice structure. We have the simple observation

**Proposition 8.2.** *If $X \to I^2X$ is an isomorphism, then the cohomological Bousfield class $\langle X^* \rangle$ is a homological Bousfield class.*

**Proof.** $\langle X^* \rangle = \langle (I(IX))^* \rangle = \langle IX \rangle$. □

Another possibility would be the slightly larger collection of all $X$ for which $\langle X \rangle = \langle I^2X \rangle$.

Perhaps the largest questions are

**Question 8.3.** Do cohomological localization functors exist? Is every cohomological Bousfield class a homological Bousfield class?

Note that an affirmative answer for the latter implies one for the former. Considering how many interesting and useful results have come from using homological localization functors, in $\mathcal{S}$ and in other stable homotopy categories, it seems that a positive answer to either of these questions would be significant. On the other hand, recall that every cohomological Bousfield class is a localizing subcategory. If there are cohomological Bousfield classes that are not homological Bousfield classes, and yet we can find interesting structure in the former, then progress will have been made in understanding the collection of localizing subcategories.

These questions can be asked in any stable homotopy category. See Section 8.2 for partial results in a Noetherian stable homotopy category. Because all the localizing subcategories are classified in the Noetherian case, the cohomological classes are probably less interesting. However, this simpler case may offer some insight into what to expect in the non-Noetherian case.

Hovey [Hov95] has computed some cohomological Bousfield classes in $\mathcal{S}$. He also conjectures that every cohomological Bousfield class is a homological Bousfield class, and shows this is true if one restricts to spectra of finite type.

8.1. **Operations on cohomological Bousfield classes.** It is possible to define various operations on cohomological Bousfield classes. Given that $\langle X\rangle = \langle IX^*\rangle$ for all objects $X$, these induce operations on the homological Bousfield classes. Here we mention only one.

Given Hovey's computations in $\mathcal{S}$, it seems there may be a large class of objects for which $\langle X^*\rangle = \langle IX\rangle$. This suggests one possible definition for a smash product operation on cohomological Bousfield classes.

$$\langle X^*\rangle \wedge \langle Y^*\rangle := \langle IX \wedge IY\rangle .$$

For those cohomological Bousfield classes that are homological Bousfield classes and $\langle X^*\rangle = \langle IX\rangle$, this obviously reduces to the ordinary smash operation on homological Bousfield classes. Likewise for those objects $X$ with $\langle X\rangle = \left\langle I^2X\right\rangle$. For arbitrary objects, however, this may not be well-defined.

8.2. **Cohomological Bousfield classes in a Noetherian stable homotopy category.** The classification of localizing subcategories in a Noetherian stable homotopy category allows us to say quite a bit about cohomological Bousfield classes. For example, we know that there is only a set of cohomological Bousfield classes, since there is only a set of localizing subcategories. This means, since there are arbitrary joins (given by the wedge) and a minimum element $\langle 0^*\rangle$, that the poset of cohomological Bousfield classes in a Noetherian stable homotopy category has the structure of a complete lattice. In this section, let $\mathcal{C}$ be a Noetherian stable homotopy category with $R = \pi_*(S^0)$, such that $\langle K(\mathfrak{p})\rangle$ is a minimal Bousfield class for all $\mathfrak{p} \in \mathsf{Spec\,R}$.

**Proposition 8.4.** $\langle K(\mathfrak{p})\rangle = \langle K(\mathfrak{p})^*\rangle$ *for all* $\mathfrak{p} \in \mathsf{Spec\,R}$.

**Proof.** First we show that $\langle K(\mathfrak{p})\rangle \leq \langle K(\mathfrak{p})^*\rangle$ for all $\mathfrak{p}$. From the classification of localizing subcategories in $\mathcal{C}$, we know that

$$\langle K(\mathfrak{p})^*\rangle = \left\langle \bigvee_{\mathfrak{q}\in S} K(\mathfrak{q}) \right\rangle ,$$

for some $S \subseteq \mathsf{Spec\,R}$. We must show that $\mathfrak{p} \in S$. Recall that $K(\mathfrak{p}) \wedge K(\mathfrak{q}) = 0$ for all $\mathfrak{p} \neq \mathfrak{q}$. So if $\mathfrak{p} \notin S$, we would have

$$K(\mathfrak{p}) \wedge \bigvee_{\mathfrak{q} \in S} K(\mathfrak{q}) = \bigvee_{\mathfrak{q} \in S} (K(\mathfrak{p}) \wedge K(\mathfrak{q})) = 0.$$

This would imply that $K(\mathfrak{p}) \in \left\langle \bigvee_{\mathfrak{q} \in S} K(\mathfrak{q}) \right\rangle = \langle K(\mathfrak{p})^* \rangle$. But we know that $[K(\mathfrak{p}), K(\mathfrak{p})] \neq 0$. Therefore $\mathfrak{p} \in S$, and $\langle K(\mathfrak{p}) \rangle \leq \langle K(\mathfrak{p})^* \rangle$.

Now we will find $\operatorname{supp}\langle K(\mathfrak{p})^* \rangle$; this is the subset of $\mathsf{Spec\,R}$ corresponding to the localizing subcategory $\langle K(\mathfrak{p})^* \rangle$ under the classification. By definition, $\mathfrak{q} \in \operatorname{supp} \langle K(\mathfrak{p})^* \rangle$ if there is some $X \in \langle K(\mathfrak{p})^* \rangle$ such that $K(\mathfrak{q})_*(X) \neq 0$. We will show that, for every $\mathfrak{q} \neq \mathfrak{p}$, $K(\mathfrak{q}) \in \langle K(\mathfrak{p})^* \rangle$. Since $K(\mathfrak{q})_*(K(\mathfrak{q})) \neq 0$, this will imply that $\mathfrak{q} \in \operatorname{supp} \langle K(\mathfrak{p})^* \rangle$ for every $\mathfrak{q} \neq \mathfrak{p}$.

Take $\mathfrak{q} \neq \mathfrak{p}$. Then

$$F(K(\mathfrak{q}), K(\mathfrak{p})) = F\left(S/\mathfrak{q} \wedge S_{\mathfrak{q}},\, S/\mathfrak{p} \wedge S_{\mathfrak{p}}\right) = F\left(S_{\mathfrak{q}},\, F\left(S/\mathfrak{q},\, S/\mathfrak{p} \wedge S_{\mathfrak{p}}\right)\right).$$

Because $R$ is Noetherian, we can choose generators $\mathfrak{q} = (q_1, ..., q_n)$. Consider the cofiber sequences

$$S^0 \xrightarrow{q_i} S^0 \to S/q_i,$$

and recall that $S/\mathfrak{q} = S/q_1 \wedge \cdots \wedge S/q_n$. Apply the exact (up to sign) functor $F(-, S/\mathfrak{p} \wedge S_{\mathfrak{p}})$ to get the triangles

$$F\left(S^0, S/\mathfrak{p} \wedge S_{\mathfrak{p}}\right) \xleftarrow{q_i} F\left(S^0, S/\mathfrak{p} \wedge S_{\mathfrak{p}}\right) \leftarrow F\left(S/q_i, S/\mathfrak{p} \wedge S_{\mathfrak{p}}\right).$$

Now, $F(S^0, X) = X$ for all $X$, and since $\mathfrak{q} \neq \mathfrak{p}$, the map $S/\mathfrak{p} \wedge S_{\mathfrak{p}} \xleftarrow{q_i} S/\mathfrak{p} \wedge S_{\mathfrak{p}}$ is an equivalence for all $i$. Therefore $F\left(S/q_i,\, S/\mathfrak{p} \wedge S_{\mathfrak{p}}\right) = 0$ for all $i$, and in particular

$$\begin{aligned} F\left(S/\mathfrak{q},\, S/\mathfrak{p} \wedge S_{\mathfrak{p}}\right) &= F\left(S/q_1 \wedge \cdots \wedge S/q_n, S/\mathfrak{p} \wedge S_{\mathfrak{p}}\right) \\ &= F\left(S/q_1 \wedge \cdots \wedge S/q_{n-1}, F\left(S/q_n,\, S/\mathfrak{p} \wedge S_{\mathfrak{p}}\right)\right) = 0. \end{aligned}$$

From the above description of $F(K(\mathfrak{q}), K(\mathfrak{p}))$ we see that $F(K(\mathfrak{q}), K(\mathfrak{p})) = 0$. Therefore $K(\mathfrak{q}) \in \langle K(\mathfrak{p})^* \rangle$.

It remains to see if $\mathfrak{p}$ is in $\operatorname{supp}\langle K(\mathfrak{p})^* \rangle$. It is not. For all $X \in \langle K(\mathfrak{p})^* \rangle$, $X \in \langle K(\mathfrak{p}) \rangle$ because $\langle K(\mathfrak{p}) \rangle \leq \langle K(\mathfrak{p})^* \rangle$. Thus $K(\mathfrak{p})_*(X) = 0$ for all $X \in \langle K(\mathfrak{p})^* \rangle$, so $\mathfrak{p} \notin \operatorname{supp} \langle K(\mathfrak{p})^* \rangle$.

We have shown that $\operatorname{supp}\langle K(\mathfrak{p})^* \rangle = \mathsf{Spec\,R} \setminus \{\mathfrak{p}\}$. The classification theorem gives a bijection, and the localizing subcategory corresponding to $T = \mathsf{Spec\,R} \setminus \{\mathfrak{p}\}$ is

$$\left\langle \coprod_{\mathfrak{q} \notin T} K(\mathfrak{q}) \right\rangle = \langle K(\mathfrak{p}) \rangle.$$

Therefore $\langle K(\mathfrak{p})^* \rangle = \langle K(\mathfrak{p}) \rangle$. □

The Bousfield lattice of $\mathcal{C}$ is a Boolean algebra on the classes $\langle K(\mathfrak{p}) \rangle$, so every object $X$ has $\langle X \rangle = \langle \coprod_T K(\mathfrak{p}) \rangle$ for some subset $T \subseteq \mathsf{Spec\,R}$.

**Proposition 8.5.** *If $X$ has $\langle X \rangle = \langle \coprod_T K(\mathfrak{p}) \rangle$ with $|T| < \infty$, then $\langle X \rangle = \langle X^* \rangle$.*

**Proof.** Using the previous proposition, we see that

$$\langle X \rangle = \left\langle \coprod_T K(\mathfrak{p}) \right\rangle = \bigvee_T \langle K(\mathfrak{p}) \rangle = \bigvee_T \langle K(\mathfrak{p})^* \rangle = \left\langle \left( \bigvee_T K(\mathfrak{q}) \right)^* \right\rangle = \langle X^* \rangle.$$

□

In the case where $|T| = \infty$, we must be careful because an arbitrary join of cohomological Bousfield classes is given by a product, not a coproduct:

$$\bigvee_{\alpha} \langle X_\alpha^* \rangle = \left\langle \left( \prod_{\alpha} X_\alpha \right)^* \right\rangle.$$

We can at least say

**Proposition 8.6.** *If $X$ has $X = \coprod_T K(\mathfrak{p})$ with $|T| = \infty$, then $\langle X\rangle \cap \mathcal{F} = \langle X^*\rangle \cap \mathcal{F}$.*

**Proof.** On the one hand, we have $\langle X^* \rangle = \{W \mid F(W, \coprod K(\mathfrak{p})) = 0\}$.

On the other hand, since $\vee$ is the join on cohomological Bousfield classes,

$$\begin{aligned}\langle X\rangle = \left\langle \bigvee K(\mathfrak{p}) \right\rangle &= \bigvee \langle K(\mathfrak{p})\rangle = \bigvee \langle K(\mathfrak{p})^* \rangle \\ &= \{W \mid F(W, K(\mathfrak{p})) = 0 \text{ for all } \mathfrak{q} \in T\} \\ &= \left\{W \mid \coprod F(W, K(\mathfrak{p})) = 0 \right\}.\end{aligned}$$

The finite objects are precisely the small objects. Therefore $\langle X\rangle \cap \mathcal{F} = \langle X^*\rangle \cap \mathcal{F}$.
□

Any collection $\langle X\rangle \cap \mathcal{F}$ or $\langle X^*\rangle \cap \mathcal{F}$ is a thick subcategory of finite objects, and so the classification theorem may allow us to say more about this situation. We also have the following partial result.

**Proposition 8.7.** *If $X = \coprod_T K(\mathfrak{p})$ with $|T| = \infty$, then $\langle X\rangle \geq \langle X^*\rangle$.*

**Proof.** We start by showing that $W \in \left\langle (\prod K(\mathfrak{p}))^* \right\rangle$ implies $W \in \left\langle (\coprod K(\mathfrak{p}))^* \right\rangle$. Let $K$ be the fiber of the natural map $\iota : \coprod K(\mathfrak{p}) \to \prod K(\mathfrak{p})$. Then after applying the exact functor $F(W, -)$ and then $\pi_*(-)$, we get a long exact sequence of abelian groups

$$\cdots \pi_n (F(W,K)) \to \pi_n \left(F\left(W, \coprod K(\mathfrak{p})\right)\right) \xrightarrow{\iota_*} \pi_n \left(F\left(W, \prod K(\mathfrak{p})\right)\right) \to \cdots.$$

This is just the long exact sequence

$$\cdots [W,K]_n \to [W, \coprod K(\mathfrak{p})]_n \xrightarrow{\iota_*} [W, \prod K(\mathfrak{p})]_n \to \cdots.$$

We claim that $\iota_*$ is injective for all $W$ and all $n$. Consider the full subcategory

$$\mathcal{A} = \left\{ X \text{ such that } [X, \coprod K(\mathfrak{p})]_n \xrightarrow{\iota_*} [X, \prod K(\mathfrak{p})]_n \text{ is injective for all n} \right\}.$$

Now, $S^0 \in \mathcal{A}$, because $S^0$ is small and this map is the inclusion $\coprod \pi_n(K(\mathfrak{p})) \hookrightarrow \prod \pi_n(K(\mathfrak{p}))$. The subcategory $\mathcal{A}$ is closed under triangles, which we see by applying the functors $[-, \coprod K(\mathfrak{p})]$ and $[-, \prod K(\mathfrak{p})]$ to a triangle and taking the kernel of the resulting map of exact chain complexes. Also, $\mathcal{A}$ is closed under arbitrary coproducts, because in $\mathfrak{Ab}$ the product of injective maps is injective. This implies that $\mathsf{loc}\left\langle S^0 \right\rangle \subseteq \mathcal{A}$, but in a stable homotopy category every object can be built from $S^0$, so $\mathcal{A}$ is the entire category.

Now, since $i_*$ is injective in the first long exact sequence above, for all $W$ and $n$, whenever $\pi_n (F(W, \prod K(\mathfrak{p}))) = 0$ we must have $\pi_n (F(W, \coprod K(\mathfrak{p})))$. Thus $W \in \left\langle (\prod K(\mathfrak{p}))^* \right\rangle$ implies $W \in \left\langle (\coprod K(\mathfrak{p}))^* \right\rangle$.

It follows that

$$\langle X^* \rangle = \left\langle \left( \coprod K(\mathfrak{p}) \right)^* \right\rangle \leq \left\langle \left( \prod K(\mathfrak{p}) \right)^* \right\rangle = \bigvee \langle K(\mathfrak{p})^* \rangle = \bigvee \langle K(\mathfrak{p}) \rangle = \langle X \rangle .$$

□

Each $K(\mathfrak{p})$ is a ring object. Hovey shows that in $\mathcal{S}$, the following ring objects all satisfy $\langle R \rangle = \langle R^* \rangle$:

$$K(n) \text{ and } E(n) \text{ for } 0 \leq n \leq \infty,\ K,\ KO,\ KT, \text{ and } Ell.$$

**Question 8.8.** In an arbitrary stable homotopy category, is $\langle R \rangle = \langle R^* \rangle$ for every ring object $R$?

Lemma 1.4 in [Hov95] shows that in $\mathcal{S}$, $\langle R \rangle \geq \langle R^* \rangle$ for every ring object $R$. The proof of this is formal, and carries over to an arbitrary stable homotopy category.

Hovey gives the example in $\mathcal{S}$, with $X = E(n)$ and $Y = E_n$, to show that $\langle X \rangle = \langle Y \rangle$ does not necessarily imply that $\langle X^* \rangle = \langle Y^* \rangle$. This is not a surprise; at first glance there doesn't seem any reason to expect that equivalent homological localizations should yield equivalent cohomological localizations. However, in a Noetherian stable homotopy category this is the case.

**Proposition 8.9.** *For any objects $X$ and $Y$ in a Noetherian stable homotopy category $\mathcal{C}$, $\langle X \rangle = \langle Y \rangle$ implies $\langle X^* \rangle = \langle Y^* \rangle$.*

**Proof.** Fix an object $X$, and consider the full subcategory

$$\mathcal{A} = \{ Z \text{ such that } \langle Z^* \rangle \leq \langle X^* \rangle \} .$$

We claim that this is a localizing subcategory of $\mathcal{C}$. If $Z_1 \to Z_2 \to Z_3$ is a cofiber sequence, with $Z_1, Z_2 \in \mathcal{A}$, then from Lemma 8.1 we get

$$\langle Z_2^* \rangle \leq \langle Z_1^* \rangle \vee \langle Z_3^* \rangle \leq \langle X^* \rangle \vee \langle X^* \rangle = \langle X^* \rangle .$$

If $Z_\alpha \in \mathcal{A}$ for all $\alpha$, then as in the proof of the last proposition,

$$\left\langle \left( \coprod Z_\alpha \right)^* \right\rangle \leq \left\langle \left( \prod Z_\alpha \right)^* \right\rangle = \bigvee_\alpha \langle Z_\alpha^* \rangle \leq \langle X^* \rangle .$$

Thus $\mathcal{A}$ is closed under triangles and coproducts, and so is a localizing subcategory. Note that of course $X \in \mathcal{A}$. Therefore $\mathsf{loc}\,\langle X \rangle \subseteq \mathcal{A}$.

Because every localizing subcategory is a Bousfield class in a Noetherian stable homotopy category, Proposition 4.7 implies that for any $A$ and $B$ we have $\langle A \rangle \leq \langle B \rangle$ if and only if $A \in \mathsf{loc}\,\langle B \rangle$. Therefore $\langle X \rangle = \langle Y \rangle$ implies that $Y \in \mathsf{loc}\,\langle X \rangle \subseteq \mathcal{A}$, and so $\langle Y^* \rangle \leq \langle X^* \rangle$. The same argument works with $X$ and $Y$ switched, and so by symmetry $\langle X^* \rangle = \langle Y^* \rangle$. □

This allows us to strengthen Propositions 8.6 and 8.7 to say that

**Corollary 8.10.** *For all $X$ in $\mathcal{C}$, $\langle X \rangle \geq \langle X^* \rangle$ and $\langle X \rangle \cap \mathcal{F} = \langle X^* \rangle \cap \mathcal{F}$.*

**Proof.** Every $X$ has $\langle X \rangle = \langle \coprod_T K(\mathfrak{p}) \rangle$ for some $T \subseteq \mathsf{Spec\,R}$, and so $\langle X^* \rangle = \left\langle (\coprod_T K(\mathfrak{p}))^* \right\rangle$. Thus for example Proposition 8.7 implies

$$\langle X \rangle = \left\langle \coprod_T K(\mathfrak{p}) \right\rangle \geq \left\langle \left( \coprod_T K(\mathfrak{p}) \right)^* \right\rangle = \langle X^* \rangle .$$

□

8.3. **Cohomological Bousfield classes in $D(\Lambda)$.** Since the objects $\Lambda(S)$ and $I(S)$ are locally finite dimensional for all $S \subseteq \mathbb{N}$, they satisfy Proposition 8.2. This has the interesting consequence

**Proposition 8.11.** *Let $S, T \subseteq \mathbb{N}$.*
(1) $\langle \Lambda(S) \rangle = \langle \Lambda(T) \rangle$ *if and only if* $\langle \Lambda(S)^* \rangle = \langle \Lambda(T)^* \rangle$.
(2) $\langle I(S) \rangle = \langle I(T) \rangle$ *if and only if* $\langle I(S)^* \rangle = \langle I(T)^* \rangle$.

**Proof.** Proposition 8.2 says that $\langle \Lambda(S)^* \rangle = \langle I(S) \rangle$ for all $S \subseteq \mathbb{N}$. Then from Propositions 6.14 and 6.15 we get

$$\langle \Lambda(S) \rangle = \langle \Lambda(T) \rangle \Longleftrightarrow S \sim T \Longleftrightarrow \langle I(S) \rangle = \langle I(T) \rangle \Longleftrightarrow \langle \Lambda(S)^* \rangle = \langle \Lambda(T)^* \rangle .$$

The second statement follows by the same reasoning, because $\langle I(S)^* \rangle = \langle \Lambda(S) \rangle$. □

The next step here would be to compute the Brown-Comenetz dual of the objects $M_{A,B,C} = \Lambda(A) \otimes k(B) \otimes I(C)$, and try to extend the above result to the collection $\mathbf{M}$ of all direct sums of $M_{A,B,C}$'s.

## 9. Other questions

The partial results given above for cohomological Bousfield classes, using Brown-Comenetz duality, seem promising. There are many other questions that can be asked about Bousfield classes in non-Noetherian stable homotopy categories, and the connection between $\mathcal{S}$ and $D(\Lambda)$. In this section I'll mention a few different directions I'd like to pursue.

9.1. **Cohomological localizations without large-cardinal axioms.** Casacuberta, Scevenels, and Smith [CSS05] were able to construct cohomological localizations in $\mathcal{S}$, but their result hinges on accepting a large-cardinal axiom (Vopěnka's principle) that is by all indications independent of the ZFC axioms. Their proofs are for the most part formal, using properties of $\mathcal{S}$ that are satisfied in an arbitrary stable homotopy category. They need the additional axiom because $\mathcal{S}$ is so big. It seems that their proofs may be transferred and altered to apply in certain other stable homotopy categories, like the derived category of a ring, or a Noetherian stable homotopy category.

9.2. **Constructing $\Lambda(S)$ and $I(S)$ objects in $\mathcal{S}$.** For a finite $S = \{n_1, n_2, ..., n_r\} \subseteq \mathbb{N}$, $\Lambda(S)$ in $D(\Lambda)$ is the finite smash of objects $\Lambda(\{n_i\})$; when $S$ is infinite, we construct $\Lambda(S)$ from the $\Lambda(n_i)$ via a colimit. Each $\Lambda(\{n_i\})$ is the cofiber of the map $\Lambda \xrightarrow{x_{n_i}} \Lambda$ in $D(\Lambda)$, where we think of $x_{n_i} \in \Lambda$ as an element of $\pi_*(S^0) = \Lambda$.

Thus the $\Lambda(S)$ and $I(S)$ are built in $D(\Lambda)$ starting with some elements of $\pi_*(S^0)$ and using operations that hold in any stable homotopy category with a notion of Brown-Comenetz duality. We might consider building analogous objects in $\mathcal{S}$, and seeing if some of the results in $D(\Lambda)$ can be transferred to $\mathcal{S}$. For example, this may be a way to show that there are objects in $\mathcal{S}$ with arbitrarily high smash-nilpotence height.

9.3. **Take $\Lambda$ as an algebra over $\mathbb{Z}$ or $\mathbb{Z}_{(p)}$, rather than a field.** The ring $\Lambda$ bears some similarity to the homotopy groups $\pi_*(S^0)$ of the sphere spectrum in $\mathcal{S}$. But $\pi_*(S^0)$ in $\mathcal{S}$ has no vector space structure, and $\pi_0(S^0)$ is $\mathbb{Z}$, or $\mathbb{Z}_{(p)}$ if we are working $p$-locally. We could tighten the analogy by taking

$$\Lambda' = \mathbb{Z}[x_1, x_2, x_3, ...]/\,(x_i^{n_i} \text{ for all } i)\,, \text{ or } \Lambda' = \mathbb{Z}_{(p)}[x_1, x_2, x_3, ...]/\,(x_i^{n_i} \text{ for all } i)\,.$$

We could then ask to what extent results in [DP08] for $D(\Lambda)$ can be transferred to results about $D(\Lambda')$, or what new results about $D(\Lambda')$ can be transferred to $\mathcal{S}$.

9.4. **Relationship between $\mathcal{C}$ and $D(\pi_*(S^0))$.** Given a stable homotopy category $\mathcal{C}$, the ring of homotopy groups of the sphere object $\pi_*(S^0)$ is an important algebraic object. We wonder if there is any connection between $\mathcal{C}$ and $D(\pi_*(S^0))$. For example, if $\pi_*(S^0)$ is Noetherian, then we know that both $\mathcal{C}$ and $D(\pi_*(S^0))$ are Noetherian stable homotopy categories, with the prime spectrum of the same ring $\pi_*(S^0)$ giving strong classification theorems in both. It seems reasonable that there may be some nice connection between the subcategories of $\mathcal{C}$ and $D(\pi_*(S^0))$.

As another example, suppose that $\mathcal{C}$ has $\pi_*(S^0) \cong \Lambda$. Then $D(\pi_*(S^0)) = D(\Lambda)$. Given what is known about $D(\Lambda)$, can anything interesting be said about $\mathcal{C}$?

## References


[Ada74] J.F.Adams, S*table homotopy and generalised homology*, The University of Chicago Press, 1974.

[BCR97] D.J.Benson, J.F. Carlson, and J.Rickard, *Thick subcategories of the stable module caetegory*, Fund. Math., 153(1):59-80, 1997.

[Bou79a] A.K.Bousfield, *The Boolean algebra of spectra*, Comment. Math. Helv. **54** (1979), 368-377.

[Bou79b] A.K.Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257-281.

[CSS05] C. Casacuberta, D. Scevenels, and J. Smith, *Implications of large-cardinal principles in homotopical localization*, Adv. Math. **197** (2005) 120-139.

[Che06] Sunil K. Chebolu, *Refining thick subcategory theorems*, Fund. Math. **189** (2006), 61 - 97.

[DHS88] E.S. Devinatz, M.J.Hopkins, and J.H.Smith, *Nilpotence and stable homotopy theory*, Ann. of Math. (2) **128** (1988), 207-241.

[DP01] W. G. Dwyer and J.H. Palmieri, *Ohkawa's theorem: there is a set of Bousfield classes*, Proc. Amer. Math. Soc. **129** (2001), no. 3, 881–886.

[DP08] W.G.Dwyer and J.H.Palmieri, *The Bousfield lattice for truncated polynomial algebras*, Homology, Homotopy, and Applications. 10(1) (2008), 413-436.

[GZ67] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Berlin, Heidelberg, New York: Springer-Verlag, 1967.

[HS98] M.J. Hopkins and J.H.Smith, *Nilpotence and stable homotopy theory II*, Ann. of Math. (2) **148** (1998), no.1, 1-49.

[Hov95] M. Hovey, *Cohomological Bousfield classes*, J. Pure Appl. Algebra **103** (1995), pp. 45-59 MR96g:55008

[HP99] M. Hovey and J.H.Palmieri, The structure of the Bousfield lattice, Homotopy Invariant Algebraic Structures (J.-P. Meyer, J. Morava, and W.S. Wilson, eds.), Contemp. Math., vol. 239, Amer. Math.Soc., Providence, RI, 1999, pp. 175-196.

[HPS97] M.Hovey, J.H.Palmieri, and N.P.Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math.Soc. **128** (1997), no. 610, x+114. MR 98a:55017

[Mar83] H.R.Margolis, *Spectra and the Steenrod algebra*, North-Holland Publishing Co., Amsterdam-New York, 1983.

[Mit85] Stephen A. Mitchell, *Finite complexes with $A(n)$-free cohomology.* Topology, 24(2): 227-246, 1985.

[Nee92] A. Neeman, *The chromatic tower for D(R)*, Topology **31** (1992), no.3, 519-532.

[Nee95] A. Neeman, *On a theorem of Brown and Adams*, Topology **36** (1997), no. 3, 619-645.

[Ohk89] T.Ohkawa, *The injective hull of homotopy types with respect to generalized homology functors*, Hiroshima Math. J. **19** (1989), 631-639.

[Tho97] R. W. Thomason, *The classification of triangulated subcategories*, Compositio Math., 105(1):1-27, 1997.

[Ver77] J.-L. Verdier, “Categories derivees”. In *SGA 4 1/2*. Lecture Notes in Math. No.569. Berlin, Heidelberg, New York: Springer-Verlag, 1977.

Department of Mathematics, University of Washington, Box 354350, Seattle, Washington 98195

*E-mail address*: lwolcott@math.washington.edu