What is Multivariable Calculus?

a long answer

Single-variable calculus asks certain questions about single-variable functions, like $f(x) = x^2$. One important question: at a certain point along the graph, how fast is the function changing? This is equivalent to asking: if I were to draw a tangent line to the graph, at that specific point, how steep would it be? This is called the derivative of the function at that point.

Another important question: If I pick two x-values, say -1 and 1, and fill in the area below my function graph, above the x-axis, and between -1 and 1, what is that area? This is called the integral of the function between -1 and 1.

Derivatives and integrals - this is the core of a calculus class. Every other question that gets asked about graphs of functions, in a calculus class, can be phrased in terms of derivatives and integrals. Different areas of math are distinguished by (1) what mathematical objects they look at, and (2) what questions they ask about those objects. In calculus, we look at graphs of functions, and ask questions about derivatives and integrals.

The most important idea in calculus, actually called the Fundamental Theorem of Calculus, is that integrals are sort of "anti-derivatives" - that integrals and derivatives are inverse operations. If you start with a function, say $f(x) = x^2$, and take its indefinite integral, you get $x^3/3 + C$, where C can be any constant. If you then take the derivative of $x^3/3 + C$, you get x^2 back.

The Fundamental Theorem of Calculus is actually stated in terms of definite integrals, and says

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

The left-hand side says: start with a function $F(x)$, take its derivative to get $F'(x)$. Then integrate $F'(x)$ from $x = a$ to $x = b$. This is equal to $F(b) - F(a)$, in words: the difference between the value of the original function $F(x)$ at $x = b$, and its value at $x = a$.

What's neat about this theorem is that it says: regardless of what happens with F between a and b, *the integral of its derivative is determined by its values at the boundary* of the interval from $x = a$ to $x = b$. The most fascinating part of multivariable calculus is how this theorem generalizes to higher dimensions.

Multivariable calculus investigates derivatives and integrals, but of multivariable functions, meaning functions that involve more than one input number. The simplest example would be a function that inputs two values and outputs one. For example, $f(x, y) = 2x - 3y^2$. If you tell me an x and a y, I can tell you the value of the function, by performing the calculation $2x - 3y^2$. Two inputs and one output means three variables total. If we let $z = f(x, y)$, then we can see the graph of the function in three-dimensional xyz-space. I always visualize these functions as height functions - sort of like the surface of a mountain. At a latitude and longitude value (some x and y, i.e. some point in the xy-plane), the mountain has a well-defined height, the

z-value. The function $2x - 3y^2$ looks like an upside-down playground slide that goes on forever.

The hardest part of a multivariable calculus class is getting used to visualizing these shapes in 3D. But it's also the most fun, because after some practice you can often truly *see* what's going on. It's not as easy to see graphs of functions of more variables, like $w = f(x, y, z) = 2x - 3y^2 + z$, which has three inputs and one output. You would need to be able to see in 4D. I think of such functions as temperature functions - each point in xyz-space has a temperature value, determined by the function. But this doesn't work with a function with 4 inputs...

Another type of multivariable function has more than one output value. For example, consider the function

$$\vec{F}(x, y, z) = (x + y,\ yz^2,\ -5xy + z).$$

If I input $(x, y, z) = (1, 2, 3)$, the output is

$$\vec{F}(1, 2, 3) = (1 + 2, 2(3)^2, -5(1)(2) + (3)) = (3, 18, -7).$$

This is called a vector function, and we usually visualize this function as assigning a 3D arrow at each point in xyz-space. At the point $(1, 2, 3)$, the arrow starts there and has its head at $(4, 20, -4)$ - which is 3 in the x-direction, 18 in the y-direction, and -7 in the z-direction. I think of 3D vector functions like the flow of wind or a fluid - at each point, there is a direction and magnitude to the wind.

In these higher dimensions, we have to make sense of what a derivative and an integral are, or should be, or what we want them to be. Think of a mountain surface in 3D. Instead of a tangent line at a point, now we have a tangent plane at each point. Instead of asking "how fast is it changing?" we ask "how fast is it changing if I move in such and such a direction?" Instead of talking about areas below the graph, between two x-values, a (volume) integral is defined to be the volume below the graph, constrained by some 2D domain in the xy-plane. But there's also a line integral, which is the area of the curtain-like sheet I get if I start with a 1D curve in the xy-plane, and pull it up to meet the graph. And there's a different kind of line integral, where we integrate a vector function instead of a height function. Again, the good news, and what makes multivariable calculus satisfying and fun, is that all the new types of derivatives and integrals that we do, are natural generalizations of the single-variable versions, and can often be drawn and visualized without too much trouble.

Without defining what these different derivatives and integrals are, I'd like to try to give you a taste of how the Fundamental Theorem of (Single-variable) Calculus generalizes to higher dimensions.

First, imagine a 1D curve in 3D, like a piece of string. Call it C, and suppose it starts at the point A and ends at B. Suppose there is a function $f(x, y, z)$, assigning a value to each point (like a temperature function). One sensible type of derivative we can take is called the "gradient" of f, and denoted ∇f. The Fundamental Theorem of Line Integrals states that

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A).$$

Just like in the single-variable case, this theorem says that the integral of a derivative is determined by its values at the boundary. If we were to move the string a little, but keep its start and endpoints at A and B, then the line integral of ∇f over the new curve would stay the same.

Here's another taste. Suppose E is some bounded 3D region in xyz-space, for example a cube or a sphere, and suppose S is its 2D boundary (the box surrounding the cube, or the surface of the sphere). Suppose $\vec{F}$ is a 3D vector field, like the flow of some fluid through space. One sensible integral is the "flux" integral of $\vec{F}$ through S - it is a type of surface integral that measures the flow of the fluid across the surface S. The "divergence" of a vector field, written "div $\vec{F}$," is a type of multivariable derivative. Then the Divergence Theorem says

$$\iiint_E \operatorname{div}\vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}.$$

The integral on the right is a flux integral. In words, the Divergence Theorem says that the volume integral of the derivative div $\vec{F}$, over the whole volume E, is determined by the flux integral of the original function $\vec{F}$ along the boundary S of E. Again, the integral of a derivative is determined by its values (in this case, an integral itself) along the boundary.

Neat, isn't it? These theorems are the culmination of an introductory multivariable calculus class, along with Green's Theorem and Stokes' Theorem, which are two other, different ways of generalizing the Fundamental Theorem of Calculus to higher dimensions.

Luke Wolcott
December 2011
www.forthelukeofmath.com