What is Multivariable Calculus?

a short answer

First let me say something about *single*-variable calculus, which you learn before multivariable calculus. A single-variable function is a rule with one number as input and one as output. For example, $f(x) = x^2$ is a rule that says, given an input x (e.g. $x = -2$), output x squared (which would be 4). So $f(-2) = 4$. If you let y be the output, you can graph such functions in the two-dimensional xy-plane. In this example, the graph of $y = x^2$ is a parabola.

Single-variable calculus asks certain questions about such functions. One important question: at a certain point along the graph, how fast is the function changing? This is equivalent to asking: if I were to draw a tangent line to the graph, at that specific point, how steep would it be? This is called the derivative of the function at that point.

Another important question: If I pick two x-values, say -1 and 1, and fill in the area below my function graph, above the x-axis, and between -1 and 1, what is that area? This is called the integral of the function between -1 and 1.

Derivatives and integrals - this is the core of a calculus class. Every other question that gets asked about graphs of functions, in a calculus class, can be phrased in terms of derivatives and integrals. Different areas of math are distinguished by (1) what mathematical objects they look at, and (2) what questions they ask about those objects. In calculus, we look at graphs of functions, and ask questions about derivatives and integrals.

Multivariable calculus investigates derivatives and integrals, but of multivariable functions, meaning functions that involve more than one input number. The simplest example would be a function that inputs two values and outputs one. For example, $f(x, y) = 2x-3y^2$. If you tell me an x and a y, I can tell you the value of the function, by performing the calculation $2x - 3y^2$. Two inputs and one output means three variables total. If we let $z = f(x, y)$, then we can see the graph of the function in three-dimensional xyz-space. I always visualize these functions as height functions - sort of like the surface of a mountain. At a latitude and longitude value (some x and y, i.e. some point in the xy-plane), the mountain has a well-defined height, the z-value. The function $2x - 3y^2$ looks like an upside-down playground slide that goes on forever.

The hardest part of a multivariable calculus class is getting used to visualizing these shapes in 3D. But it's also the most fun, because after some practice you can often truly *see* what's going on. It's not as easy to see graphs of functions of more variables, like $w = f(x, y, z) = 2x - 3y^2 + z$, which has three inputs and one output. You would need to be able to see in 4D. I think of such functions as temperature functions - each point in xyz-space has a temperature value, determined by the function. But this doesn't work with a function with 4 inputs...

In these higher dimensions, we have to make sense of what a derivative and an integral are, or should be, or what we want them to be. Think of a mountain surface in 3D. Instead of a tangent line at a point, now we have a tangent plane at each point. Instead of asking "how fast is it changing?" we ask "how fast is it changing if I move in such and such a direction?" Instead of talking about areas below the graph, between two x-values, a (volume) integral is defined to be the volume below the graph, constrained by some 2D domain in the xy-plane. But there's also a line integral, which is the area of the curtain-like sheet I get if I start with a 1D curve in the xy-plane, and pull it up to meet the graph. Again, the good news, and what makes multivariable calculus satisfying and fun, is that all the new types of derivatives and integrals that we do, are natural generalizations of the single-variable versions, and can often be drawn and visualized without too much trouble.

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