THE PHENOMENOLOGY OF MATHEMATICAL UNDERSTANDING

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Abstract

We present the results of a phenomenological methodology that allowed for the investigation of the experience of understanding an abstract mathematical object as effectively lived by active mathematicians. Our method of analysis reveals the essential structure of such phenomenon and, as a consequence, permits us to address the *conditions of possibility* for the occurrence of this particular phenomenon. We show that the different modalities of the experience of understanding an abstract mathematical object, as unearthed by the elucidation of the three phenomenological classes (parts and wholes, identity in a manifold, and presences and absences), reflect – in addition to well-expected mathematical functional aspects – what we call an *embodiment of abstraction*.

(…) in order to be able to do mathematics you do not have to ask yourself of what earthly use it is to you. The usefulness of mathematics increases the *pulsion* but not dexterity; the true usefulness happens from the pulsation between reality and aesthetic sense. The last thesis says that dexterity is not talent or genius. Everyone can do mathematics like everyone can ride a cycle. The main injunction here is just: do it!

Mathematical Pulsation at the root of invention, René Guitart.

Introduction

G.-C. Rota, a mathematician who was deeply interested in phenomenology and studied Husserl at length, appealed for a "*realistic description of what is going on in mathematics*" (Rota, 1997a), for he was concerned with the loss of contact of some philosophical concepts with the effective world of mathematical practice (Rota, 2006). We find in (Hersh, 1999; Hersh, 2014) similar concerns for there is, in Hersh's opinion, a too loose connection between the philosophy of mathematics and the actual way mathematics is done. In recent explorations grounded in historical investigations, case studies, and the social dimension of mathematical practice, we find proposals for different ways to think philosophically about mathematics (Corfield, 2004), but also to reconsider and nuance the actual distinction between mathematical concepts and mathematical intuition that tends to state that mathematics has become more abstract and detached from intuition than it was in Euclid's time (Rodin, 2010). When we consider the number of famous mathematicians who have given insightful details on what it is mathematicians accomplish, what the phenomenology of mathematical proof and mathematical beauty are, or what it is to deal with abstract matter (Gowers, 2002; Thurston, 1994; Rota, 1997a; Rota, 1997b; Halmos, 1973), we can infer a significant concern and desire for real dialogue between mathematicians and philosophers. As stated by Hersh, there is a real necessity to consider the lived experience of active mathematicians if we were to propose a philosophy of *real* mathematics (Hersh, 2014). Recent works have shown that a historical approach based on case studies and mathematicians' testimonies about their work and practice demonstrates the inescapable role of intuition in advances in mathematics (Hersh, 2014; Rodin, 2010), and gives a refreshed definition of mathematical intuition. As Rodin shows, intuition in mathematics is subject to historical development just like mathematical concepts and is not, borrowing Hersh's wording, a direct connection to the transcendental. A recent phenomenological approach to the practice of mathematics has investigated the intuitive experience in mathematics and unfolded the essential structure of such phenomenon (Van-Quynh, 2015; Van-Quynh, 2017) that could be fruitfully put in perspective with theoretical work on mathematical intuition (Chudnoff, 2011) and with models of mathematical activity developed by great mathematicians like Hadamard and Poincaré.

The present text investigates the concept of mathematical understanding from a descriptive perspective in the aim to propose a phenomenological description of the experience of understanding an abstract (mathematical) object. What does it mean to understand mathematics? What does it feel like, from the mathematician's point of view? What are the characteristics of the experience of having an advanced understanding of a mathematical object? To address these questions, we first proceeded to interviews of mathematicians according to a specific protocol, which we will describe later on. But if we were to perform a phenomenological study, adopting a descriptive perspective does not mean that we would have limited ourselves to the mere collections of testimonies and subjective data. Indeed, in the vein of Husserl's phenomenological programme, we disclose the essential structure of the experience of understanding in mathematics by utilizing a precise methodology that consists in three main steps:

i) investigate the experience of understanding a mathematical object as effectively lived by mathematicians by collecting subjective descriptions of such an experience, using a particular interview method;

ii) make then a systematic analysis of the collected data inspired from the phenomenological tradition in order to elucidate the three formal structures of the phenomenon (parts and whole, identity in a manifold, and presences and absences);

iii) and finally, perform an eidetic variation that allows the identification of the generic (or essential) structure of the phenomenon, *i.e.* the conditions of possibility for this phenomenon to be.

The mathematical object we focus on is called a group. A group is a specific mathematical object, well-known to professional mathematicians and taking a variety of forms throughout mathematics. To examine the experience of understanding a group as effectively lived by active mathematicians, we used a method of assisted introspection in order to allow interviewed mathematicians to enter in close contact with their subjective experience of "understanding a group" and to describe it with precision. We obtained subjective descriptions of the experience that go beyond its solely mathematical aspects, unfolding its sensitive, kinesthetic and visual aspects, by considering how the phenomenon of understanding a group manifests itself to an interviewed subject and helping the subject reveal to herself aspects of the experience of which she was previously unaware.

The methodology we used is that of the elicitation interview (EI) method, a powerful tool to enrich the reflective content of the mental acts of a given nature (e. g. listening to a sound, memorizing a poem, meditating...) in allowing a subject to access the pre-reflective content of these acts. Indeed, unassisted introspection often does not suffice to provide a pin-pointed description of the cognitive processes that enter into play in a given experience. The pre-reflective dimension of an experience is often missing, or is very shallow, when a subject gives spontaneously a description of an action or an experience. The term *pre-reflective* is understood in the sense detailed by Vermersch in (Vermersch, 1994) who grounds his experiential works in E. Husserl's phenomenology. *Reflective* consciousness is distinguished from *pre-reflective* consciousness according to the following: *reflective* consciousness is what one knows by knowing one knows it, *i.e.* what one is "already aware of." *Pre-reflective* consciousness is what one knows without knowing one knows it.

EI permits, through verbalization, the revelation of implicit knowledge of a subjective experience by giving access to its pre-reflective content. With such an exploration, the natural tendency of "*speaking in general*" is avoided, as the subject is guided with the aim to focus her attention on a singular intimate experience, and to allow her to *become aware* of and describe the pre-reflective dimension of the experience with which she enters in close contact. To keep this article under a reasonable number of pages, we do not enter the details of the methodology; the reader can find extensive details on this method and its actual practice in (Vermersch, 1994; Vermersch, 1999; Petitmengin, 2009; Van-Quynh, 2017). Practically, in this study, we conducted in-depth elicitation interviews with four PhD mathematicians from a range of specialties: analysis, geometry, algebra, and mathematical physics. These mathematicians – all holding positions in university departments – were selected to capture the range of contexts in which mathematical groups are encountered. The outcome of these EIs was our experiential material that we subsequently analyzed in a procedure that we explicate later in this text.

The decision to focus on the notion of "group" was based on several factors. A group is a set with a binary operation satisfying certain axioms; in the Appendix we give more details and some examples, but for the purposes of this paper it suffices to know that groups are basic mathematical objects that appear in many areas of mathematics. Different groups have different context-specific definitions and properties, but all satisfy the definition of a group. They are basic but not too basic, and common enough that every mathematician has experience working with different groups. By virtue of their formal definition, groups are abstract, and are in fact one of the first topics introduced in a conventional "abstract mathematics" course. Compared to many other mathematical objects (*e.g.* graphs and curves, dynamical systems, geometric objects, probability distributions), groups have no straightforward physical representations in the non-mathematical world. In fact, most groups have no obvious associated mental imagery; one must either construct imagery or rely on non-visual and non-spatial mental phenomena. Developing an understanding of groups requires grappling with their abstract and non-visual nature, and this is something every professional mathematician has experience doing.

Two categories of understanding – functional and felt understanding – elucidated through two levels of understanding: basic and advanced

Our interview data classified into two broad categories of understanding: functional and felt. Functional understanding refers to a mathematician's ability to manipulate and work with a group, to be able to answer questions about it, and to have the knowledge of its context and history. For example, the functional aspects include knowing the group's elements, knowing the generators and relations, and knowing the group's representation theory[1]. Felt understanding refers to the embodied, kinesthetic, emotional, or sensuous aspects of the experience of understanding, for example mental imagery of the group or perceived relationship to it.

Descriptions of functional understanding were primarily and easily volunteered by interviewees, and align with the conventional orientation of mathematics towards problem-solving. But these are not enough for a full phenomenological description of the experience of understanding a group. Felt understanding has traditionally evaded description and analysis, but it is exactly these felt aspects that are suited to elucidation through guided introspection and the elicitation interview.

In each of these categories, we make the distinction between basic and advanced understanding. Mathematicians were first asked to focus on the experience of working with a group that they felt they understood well; these descriptions contribute to characterizing advanced understanding. But, as we will see and explain in detail in the following section, the descriptions of an advanced understanding were indirectly enriched by our second theme of interviews. Indeed, mathematicians were asked in a second instance to focus on a group that they felt they understood at a basic level or hardly at all. This is what we associated to the experience of a basic group understanding.

Addressing the generic structure for the experience of understanding a mathematical group by identifying three formal phenomenological classes

The generic structure we extracted from the subjective data will be presented following the scheme proposed by Sokolowski in his book *Introduction to phenomenology* (Sokolowski, 2000), comprising three interrelated formal structures necessary for a

[1] Throughout this paper, mathematical jargon may be taken at face value and need not be understood. An appendix provides some precise definitions and examples.

proper phenomenological analysis of a given phenomenon. Namely, the structures of:

i) parts and wholes;

ii) identity in a manifold;

iii) presences and absences.

Laying out the array of these formal structures will provide a phenomenological description for the phenomenon of *understanding a mathematical group* that is not supposed to convey the thoughts and raw descriptions of any one mathematician in particular, but that will reflect the essence of such experience after having performed an eidetic variation of the subjective descriptions we collected.

In what follows we give glimpses of these three structures in order to make, for those not familiar with them, our presentation easier to read and more accessible. The reader interested in the full details of these matters will refer to Sokolowski's book (Sokolowski, 2000 – chapter 3).

According to this author, if we are aware of the three forms it is then possible to understand what is going on for a particular phenomenon, here that of "understanding a mathematical group," and what constitutes an essential apprehension of it. The structure resulting from the symbiosis of these three forms will form the conditions of possibility of such experience and reflect the essential constituents of the generic structure we extracted from our subjective data (Zahavi, 2003; Sokolowski, 2000).

i) Parts and wholes

There exist two different kinds of parts in a whole: pieces and moments. The former are what we call *independent* parts while the latter are *nonindependent* parts. Pieces are parts that can subsist and be presented even apart from the whole; they can be detached from their wholes (Sokolowski, 2000; p.22). On the contrary, moments are parts that cannot subsist or be presented apart from the whole to which they belong; they cannot be detached. As illustration of pieces and moments, Sokolowski gives the example of a branch of a tree – this is a piece, for the branch can exist independently of the tree (a branch presents itself as an independent entity). A moment is, for example, the color of a thing, which cannot occur apart from surface or spatial expanse.

A typical characteristic of moments is that they are the kind of part that cannot become a whole. In other words, a whole being something that can exist and be experienced (Sokolowski speaks about a *concretum*), a piece is a part that can itself become a *concretum*, while moments cannot become *concreta*. Moments exist and are experienced along with other moments of the whole to which they belong: they exist only as blended with their complementary parts (ibid. pp.26-27).

In our analysis, after categorizing the functional and felt parts of the experience of understanding a group, we identified pieces and moments of each. First, in the case of functional understanding, we formed a list of items that are essential (the functional moments of the understanding): "having a sense of the group," "multiplicity and adaptability," as well as "knowing basic manipulation" and "knowing properties of the group." This list gives the "mathematical conditions of possibility" for our phenomenon. In other words, the experience of a deep understanding of a group cannot manifest if the items of this list are not all mastered by the mathematician.

The functional pieces of the experience of understanding, on the other hand, can exist outside of this experience. For example, we found that "knowing the history of the group" is a piece of understanding a group. Indeed, such a knowing can exist independently of the experience of understanding the group and moreover, knowing the history of a group does not necessarily secure/imply the full functional understanding of it. Knowing the history of a group can be a *concretum* on its own, perhaps derived from a non-technical book or course on the history of mathematics. Other pieces are "knowing generators and relations," "knowing group actions, symmetries, and representation theory," and "knowing subgroups and types of elements."

These three items are pieces, rather than moments, because their mathematical content exhibits a high level of modularity. If one knows the generators and relations of a group – at a basic or a deep level – this knowledge or clarity in most instances does not carry very far. It is possible to know a great deal about the generators and relations of a group, but still know very little else or have few means to transfer this knowledge to wider insights. As we will see, our mathematicians classify "knowing generators and relations" as indicative of only basic functional understanding, with views like, "*you can say it's the group of these letters that have these relations… that's not really to know it yet"* (Q-1). In this sense, "knowing generators and relations" – at a basic or advanced level – can exist independently as a piece of understanding a group.

Avoiding the technical details, "knowing group actions, symmetries, and representation theory" is similarly very modularized in relation to full understanding (these three are intimately related and essentially synonymous). Representation theory of groups can be considered a subfield of group theory, with its own specialized methods and results. One can work in this subfield and develop understanding that is independent of the other pieces. Furthermore, there are groups with no useful or meaningful actions or representation theory that one can nevertheless understand deeply. This shows that knowledge of group actions, etc., is not essential to understanding a group, in the same way that Sokolowski's identification of the branch as a piece of the tree is supported by the existence of branchless trees.

To "know subgroups and types of elements" of a group is, almost literally, to know pieces of the group. And here there is genuine modularity. For example, the group of matrices has a subgroup of orthogonal matrices, which is itself a group. Any understanding of the group of orthogonal matrices forms a piece of understanding of the larger group. One interviewee says, "*orthogonal matrices, or Hermitian, or symmetric… those are subsets. These subsets interact between themselves, but with the rest of the group as well. And that, that is also a part of understanding a group*" (Q-2). One can understand these substructures without understanding how they fit together into the larger structure. Moreover, some groups have no or few interesting subgroups.

In the category of felt understanding we also identified pieces and moments. As we will illustrate and discuss in later sections, mathematicians visualize streamlined and pattern-based mental imagery when working with a group that they understand well. This experience of "pattern-based mental imagery" is a piece, as is the aspect of "immediacy" that accompanies understanding. The streamlined imagery of a group could be cultivated in an independent fashion by abundant problem-solving with diagrams and drawings. By training in such a way, one would experience a shift from specific imagery to more pattern-based (one interviewee says "*archetypal*") imagery, and as a result experience a deepening of understanding of the group. But this development could occur independently from other aspects, especially functional, of full understanding. Several mathematicians report an experience of immediacy when they understand a group well; "*When I know it well, I can immediately think about it and start working with it and use it*" (Q-3). In a similar way to imagery, one could specifically and independently cultivate this immediacy of understanding.

On the other hand, descriptions of a close relationship between the mathematician and the group – "perceiver as comfortable/happy," "group perceived as a good friend, with a personality," and "relationship as familiar/harmonious" – are essential moments of the felt experience of deep understanding. These cannot exist independently from the understanding. In the same way the moments of the functional understanding were the "mathematical conditions of possibility" of our phenomenon, the latter form the "conditions of possibility" for the experience; without them present it is not possible to envision advanced understanding.

ii) Identity in manifolds

This is an important aspect for a phenomenological description of an experience. The theme of identity in manifolds is straightforward when we describe the perception of a cube, to borrow Sokolowski's example and wording: "*a cube as an identity is shown to be distinct from its sides, aspects, and profiles, and yet it is presented through them all*" (p.27). The theme of identity in manifolds can be extended to any kind of thing that can be presented to us. To take another example, the expression of a fact can be expressed in different ways that are aspects of the same fact; the different expressions of these aspects are various ways of presenting it and represent the same fact. For instance, we might express the fact that a street is covered with snow by using passive or active voice, using different languages, or with several tones of voice, but still we will be describing the same fact. Furthermore, the identical meaning can be presented through many sentences and expressions that have not yet been stated, and may never be stated.

In other words, the same fact can be expressed in a manifold of ways but this fact is just different from any and all of its expressions. The meaning is the identity that is within and yet behind all of its expressions. This is what we call *an identity in a manifold.*

In our study and analysis, the identity in a manifold was obtained through several types of multiplicity of descriptions:

a) within the statements of a single mathematician. Indeed, we could unfold that a mathematician describes her understanding of a specific group in a manifold of ways, and has many ways of trying to say the same thing. For example, one interviewee described the feeling of advanced understanding as like "…*comfort with old friends,*" and later as, "*you like what they have to offer.*"

b) across several distinct groups and types of groups. Here it is paramount to clarify that a mathematical group is a type of mathematical structure, of which there are many examples: the symmetric group, the dihedral group, the general linear group, etc. In investigating the experience of understanding mathematical groups, we have found invariants across a range of specific groups. Mathematicians were asked to pick a group that they felt they understood; some chose the symmetric group, whereas others chose the dihedral group or the general linear group. This important diversity allowed us to move beyond any specific example of understanding a specific group, to capture the experience of "understanding a mathematical group."

c) with the comparison of the interviews of different mathematicians. By proceeding with several interviews, our aim was to obtain a deeper objectivity on the same identity by including the dimension of intersubjectivity. Indeed, by doing so we obtained a richer array of manifolds than the one we would have arrived at if we had dealt with the EI outcome of a single mathematician (see (Sokolowski, 2000) p.31 on this particular matter: "*when other perceivers are brought into the picture, the same identity takes on a deeper objectivity, a richer transcendence; we now see it not only as the thing we would see differently if we were to move this way and that, but also as the very same thing that is being seen, right now, from another perspective by someone else*"). In addition, a single elicitation interview of a mathematician might not suffice (and surely does not suffice) to bring to the fore all the nuances and facets associated to the mathematician's experience of her understanding. Additional interviews of the same mathematician may have permitted to go deeper into the description of such experience by allowing the subject to be more and more in close contact with her intimate experience of understanding (Petitmengin, 2001; Petitmengin, 2013; Van-Quynh, 2015; Van-Quynh, 2017; Vermersch, 1999). In the interest of intersubjectivity, however, our data collection was limited to a single interview per mathematician. And, as phenomenology points out very often, "*The horizon of the potential and the absent surrounds the actual presences of things. The thing can always be presented in more ways than we already know; the thing will always hold more appearances in reserve*" (Sokolowski, 2000. See also for instance Giorgi, 1985; Zahavi, 2003).

iii) Presences and absences

Recall that in phenomenological terminology, to intend is to commit an act of consciousness; every such act (of intending) is directed towards an intended object. And then "*presence and absence are the objective correlates to filled and empty intentions. An empty intention is an intention that targets something that is not there, something absent, something not present to one who intends. A filled intention is one that targets something that is there, in its bodily presence, before the one who intends*" (Sokolowski, 2000 p.33). –In phenomenology, in order to properly describe an object or a phenomenon, the blends of presences and absences that belong to the object or the phenomenon in question need to be spelled out[2]. Indeed, presences and absences are related to the same object or phenomenon, as things are given in a mixture of presences and absences (or a mixture of filled and empty intentions) the same way they are given in a manifold of presentation. This means that there is an identity "behind" and "in" absences and presences and that these parts inform each other. A straightforward example is the perception of a dice: there are profiles and sides of it that we actually see and those that we don't see. The former are present, while the latter are absent. But all those parts, whether being presences or absences, belong to the same object; the dice exists itself through the identity identified across the two. And what operates in the perception of material objects, operates also in any kind of things that are presented to us (Sokolowski, 2000, pp.22-27).

The filled intentions characterized by the bodily presence of an object before the one who intends (Sokolowski's wording) are in our study evidenced with the presence of "mathematical knowledge" along with feelings that are effectively experienced by the mathematician. In other words, in the mathematician's experience of an advanced understanding, the bodily presence that is the signature of a filled intention manifests here by the presence of aspects that are there mathematically (mastered functional aspects), physically and sensuously.

The outcome of the interviews shows that mathematicians do intend the absent, especially when describing their experience of a basic understanding. The absence is effectively for them a phenomenon in itself. Indeed, mathematicians point at the different (functional and not functional) aspects that are not part of their understanding and feel the absence of them, for they feel and acknowledge the absence of certain criteria. A basic understanding is verified with their absence and makes the mathematician judge that she does not fully understand the group. For example, "*you don't have a good picture of it*" (Q-4), "*I don't have many ways to think about it*" (Q-5), and "*if you*

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[2] For the detailed discussion on these phenomenological concepts, see (Sokolowski, 2000, pp.33-40).

cannot manipulate something, that means you didn't understand it well" (Q-6).

The experience associated with a basic understanding of a group is rich in absence, and our results show how important it is to access "what is not there," "what is emptily intended." For if we had relied on the present parts only, we would have ended up with an indigent description of what it is like to not fully understand a mathematical group. Sokolowski underlines at different stages of his phenomenological analysis methodology how important the absent parts are, and how often they are neglected in philosophical descriptions and analysis[3].

We might think that a basic understanding is *obviously* characterized by absences, because if the absent were not absent, and thus were present, there would be no basic understanding and the knowledge would be experienced and verified as advanced or full. Indeed, when a mathematician appreciates her understanding, she precisely does it thanks to the functional and felt aspects that are not absent: the presence cancels the absence and the understanding is registered and thus confirmed by the mathematician as advanced. But the description of absent parts is also evidenced in the elicitation of the experience of an advanced of understanding. Indeed, we find for instance that an advanced understanding is characterized by the absence of too-accurate imagery as well as the absence of specificity of imagery. One mathematician refers to imagery of a group he understands well in describing that he sees *"something with some sides but not too many sides… not too few sides because then it's too particular*" (Q-7) and later says, "*It's an archetype of an element. It's not an element in its own*" (Q-8).

Furthermore, mathematicians have a rich "absence"-felt experience of lacking understanding. It is "*uncomfortable. I feel a bit of anxiety*" (Q-9). The math may be perceived as a stranger; "*It's like meeting a new person… I don't see him immediately as my enemy, and he may actually be some mysterious and interesting person*" (Q-10). The relationship is "*unfamiliar*", or "*it's like putting yourself in the jungle and trying to detach yourself from the trees*" (Q-11).

Elucidating both the present and absent aspects of an experience allows for a deeper analysis. Indeed, the identity of the full/advanced understanding is given across the

[3] Sokolowski emphasizes that while presences are a theme in philosophy that has received enormous attention, absences have been neglected and "replaced" by the tendency to speak about an object that is not present using images or concepts of the object, which thereby become present.

difference of presence and absence and is not given only in the present parts (recall the example of the dice). In addition, we could bring to the fore that the advanced understanding as experienced by the mathematician corresponds to a cumulative fulfillment that is the result of a list of presences that gradually leads up to the judgment of an advanced understanding. While it might seem straightforward that the functional aspects of an advanced understanding should be present as a cumulative list of functional items (knowing the generators and relations, the subgroups, the group actions; having a sense of the group, etc.), the felt aspects associated to the *presence* of several feelings or sensations put forward an embodiment of the experience associated to an advanced understanding of a mathematical group. In other words, the mathematician is not only dealing with an experience of pure thoughts but does intuit[4] a sensorial component in her experience. We read for instance, "*If you touch it a little, if you have a complete understanding of it, you sense that it is stable*" (Q-12). Therefore, the mathematician dealing with very abstract matter is involved in an experience that embraces her whole being (comfort, sensation of ease, pleasure) and for which the apprehension of the abstraction – or the apprehension of an abstract object – can be personified. As an example, one mathematician speaks about a family member, another about good friend with whom "*you like what they have to offer*" (Q-13).

The generic structure of understanding a group in mathematics

Thanks to the complementarity of the three formal structures for the analysis of our subjective data we could extract a generic structure of the phenomenon of an advanced understanding. Indeed, the identification of pieces and moments was made possible by the joined analysis of the three formal structures.

Practically it was a matter of identifying first the similarities among the different subjective experiences that reflect the typicality of the phenomenon in order to, second, reach empirical universals of the phenomenon by making an identity synthesis. The latter is no longer a search for what is similar between the different experience descriptions but for what is the "one" behind the multiplicity. The last step was then to perform a sort of eidetic variation that allowed us to extract the features of the phenomenon that it would

[4] We use here the term intuition/intuit in the following phenomenological sense: *intuition is having an object actually present to us, in contrast with having it intended in its absence*.

be inconceivable for the phenomenon to lack. We employ the expression of "*sort of* eidetic variation" for we do not really enter in the realm of imagination, as a pure eidetic variation in phenomenology would demand (Sokolowski, 2000; Zahavi, 2003; Depraz, 2012). Rather we performed a sampling eidetic variation, as we did not intend to unfold the essential insight of the understanding of a mathematical group from a single description but in taking all the individual descriptions together in a comparative view in the aim to identify an identity in manifolds (see item (c) of identity in manifolds subsection) grounded on intersubjectivity. As we have already explained in the "presences and absences" subsection, presences and absences were a means to address the moments of the phenomenon of an advanced understanding by comparing the outcomes of the EIs on a group deeply understood by a mathematician and one understood at a basic level. The generic structure we achieve then is made of a series of moments referring to functional and felts aspects that put together make the conditions of possibility for the understanding to be deep/advanced. For the sake of simplicity, we choose to display the moments according to these two categories and not to merge them. We include some representative quotations from interviews.

Functional moments:

i) Knowing properties of the group. The mathematician knows a variety of properties and characteristics of the group, and "*can compare and contrast the group with others in terms of different properties*" (Q-14). For example, "*it's to know things like, is it commutative? Or not? How many generators?... to fit it in some patterns… sort of classified with respect to others, to other characteristics*" (Q-15).

ii) Multiplicity and adaptability. The mathematician has several ways of thinking about and working with the group, and is flexible and adaptable: "*I'm familiar with how to visualize it. I know several different ways to think about its elements*" (Q-16), "*You know its insides. You know how it relates with other things.*" (Q-17), "*It's like a tool, but it's more dynamic than a tool*" (Q-18).

iii) Having a sense of the group. The mathematician has "*a good picture (not necessarily geometric in nature) of the group telling you intuitively what statements may be true and what statements should be false about the group*" (Q-19), or "*an idea of the kind of result that you'll obtain*" (Q-20).

Felt moments:

i) Perceiving mathematician feels comfort, pleasure, stability, and/or reassurance. The bodily sensation can range from a "*tingle of pleasure*" (Q-21), to a "*sense of satisfaction*" (Q-22), or an "*euphoria*" (Q-23).

ii) The mathematical object is perceived as a good friend, and given a character or personified. For example, one mathematician associates a "*discrete personality*" (Q-24) to one group and a "*character*" (Q-25) that is "*more fluid, less rigid*" to another. The group becomes "*a very good friend. Someone that you know very well*" (Q-26).

iii) Relationship between perceiver and perceived is (experienced by the perceiver as) "*familiar*" and "*harmonious.*"

Discussion

The structure we have unfolded can be put in perspective with the project that R. Hersh proposes in *Experiencing mathematics* in order to investigate the lived experience of perception of an active mathematician.

> *"I think it is possible to take, as the "foundation of mathematics", the lived experience of the active mathematician – which, indeed, so many have already described, in poetic or metaphorical language. That experience is an interaction between perceiver and perceived. Indeed, that interaction should be regarded as coming first, making possible the distinction between the perceiver and the perceived (the mind of the mathematician on the one hand, the mathematical object or entity of the other hand). The act of experience of mathematical perception, like that of visual or auditory perception, is an interaction between two interpenetrating partners – the perceiving mathematician and the perceived mathematical object or entity"* (Hersh, 2014 p.129).

Here our purpose was not to directly tackle the issue of "*what it is for a mathematician to perceive a mathematical object*" but to get an insight into the lived phenomenon of the understanding of an abstract object[5]. While the term "understand" can be very controversial and may demand careful definition, we did not enter the philosophical question of "*what it is to understand (a notion or a theme)*" but decided to ground our investigation on descriptions of "*how* it is to understand a mathematical object" given by

[5] In two remarks at the end of this document, we also discuss the irrelevance of mathematical ontology to our project.

active mathematicians who judge themselves to have an advanced understanding of a mathematical object. As specified in the Introduction, we chose the concept of group because it is common and pervasive in higher mathematics, but is abstract and non-visual. Dealing with an abstract object, we could enter into private experiences of how it is and how it feels to work with an abstract object and, as a benefit, to get an insight into the phenomenon of perception of a mathematical object. With our interviews we focused on "how it is to understand this group" and "how would you describe this understanding?" instead of centering our questioning on "why do you understand this group?" By questioning mathematicians about their understanding of a mathematical group when they are in close contact with their inner experience and by trying to address the different modalities of this understanding, we smoothly entered the realm of perception – that of a mathematician perceiving a group she believes she deeply understands. Therefore, to a certain extent our interviews were those of perceiving mathematicians who describe their experience of perception and relationship with an abstract mathematical object – what R. Hersh labels "*the interaction between the perceiving mathematician and the perceived mathematical object.*"

The investigation of the two types of experiences (advanced and basic understandings) allowed us to extract the essential aspects[6] of the phenomenon of deep understanding, and suggests a geometric metaphor: deep understanding corresponds to a *closeness* between perceiver and perceived.

Imagine a professional mathematician progressing from basic to advanced understanding. She is initially presented with a new group and is remote from it. She comes to feel that she doesn't understand this group because of functional indicators – she doesn't know many properties of the group, she doesn't have many ways to think of the group, she doesn't have a sense of the group – but also because of felt indicators. It might be the case that her mental imagery is more literal and less pattern-based, or she notices a lack of immediacy when working with the group. This mathematician will feel "*uncomfortable*" (Q-27) or "*a bit of anxiety*" or "*agitated*" (Q-28). The process is "*like meeting a new person. I don't see him immediately as my enemy, and he may actually be some mysterious and*

[6] The structure we propose here is not claimed to be exhaustive and, as with any phenomenological study, is subject to further revisions and/or modification. Indeed, what would have come out if we had interviewed more mathematicians? And what would repeated EIs of one mathematician have led to? We might, by doing so, unfold aspects that were not brought to the front here (see for instance (Depraz, 2012); (Zahavi, 2003); (Giorgi, 1985)).

interesting person" (Q-29). The relationship is "*not familiar*" (Q-30) or "*It's an old acquaintance... But it's not a relationship that's been maintained*" (Q-31).

As this mathematician begins to work with the group, the distance between perceiver and perceived becomes less. She will add to her knowledge, and begin to experience the functional indicators for deeper understanding – multiplicity and adaptability, knowing group properties, having a sense of the group. Her felt experience will also gradually shift, for instance towards pattern-based and more streamlined imagery, or towards a feeling of immediacy when working with the group. There will be a development from discomfort to comfort, towards "*satisfaction that things are in their place…You're able to instill order and begin cleaning… You've made yourself feel well in that house. There is harmony*" (Q-32). The group will no longer be "*nebulous*" (Q-33) but, despite its abstract nature, will acquire a character or personality. As the distance becomes less, the relationship becomes more like "*comfort with old friends. You know what to expect. And you like it. You like what they have to offer*" (Q-34). And the gap can shrink to the point that "*It's a part of me. Some of these things are a part of me. Even when I'm thinking of me as not a mathematician, some of these things are a part of me*" (Q-35). At this stage of advanced understanding, the distance between perceiver and perceived is small, and we indeed find phenomenological indicators of Hersh's "*interpenetrating partners*" (Hersh, 2014).

We have shown that mathematicians are able to take something so abstract and conceptual as a group, and relate to it like a friend or a family member or a part of themselves. Our evidence points to the "embodiment of abstraction" as a crucial component of advanced mathematical understanding. According to our list of functional moments, one must start from the conditions of possibility – a formal definition of the group, and a collection of facts and properties satisfied by the group – and proceed to develop a "sense of the group," "*telling you intuitively what statements may be true and what statements should be false about the group*" (Q-36). Unlike understanding a musical instrument or the workings of a motorcycle, these abstract groups cannot be seen or touched or manipulated. Yet, at the felt level abstraction is embodied in pattern-based imagery, and sensations of comfort, pleasure, or stability. Having arrived at a phenomenological description of the understanding of an abstract object, and mapped the development from basic to advanced understanding, we find that mathematicians are able to embody their abstractions, in a progressive relationship – from keeping it as a stranger, to assigning it a character/personality, becoming friends, and making it a part of themselves.

Before closing the discussion, we would like to say a few words about what is revealed by the fore-mentioned embodiment and the different modalities that characterize the phenomenon we have investigated. Husserl develops the thesis that ideal objects (*i.e.* mathematical objects) are objects that by essence cannot be given straightaway and through sensuous acts but that have no less real existence than physical objects despite their apparent and manifest a-temporality. What we might call "intellectual intuitions" never fully break their connection with the fundamental mode of simple perceptual intuition (Husserl, 1970; Husserl, 1983; Desanti, 2001 pp. 222-225; Drummond, 2015 pp.24-27). Perceptual consciousness delivers awareness of an object itself, "in the flesh", and not just an image or mere appearance of it. Nonetheless, intellectual intuitions are not reducible to simple perceptual intuitions even if rooted in them (Kidd, 2014 pp.137-139).

The outcome of our study matches these features. Indeed, when a mathematician "speaks" about or evokes an ideal object (here a group), she performs an act of intellectual perception that displays similarities with straightforward perception: the ideal object is perceived with contingency and concreteness, "has some flesh" (recall for instance "*if you touch it a little, (…), you sense that it is stable*" (Q-12)). As evidenced thanks to our interviews, the perception of the group by the mathematician displays *presentational phenomenology*. To put it shortly, as E. Chudnoff defines it, a perceptual experience has presentational phenomenology when "*one experiences a scene as if the scene's object and their features are directly before the mind*" (Chudnoff, 2011, p.631). Therefore, the mathematician is in a state of "seeing" a group that goes beyond simple visualization of it, the abstract object is personified, is *felt* by the subject; for the subject the abstract object *exists* not anymore (only) as a mere image or appearance.

But, and this to challenge on the thesis that intellectual intuitions are not reducible to simple perceptual intuitions, the perception of the abstract object involves an act of *conceptual intuition* of an exact essence. Indeed, following Drummond's understanding of the conceptual intuition of exact essences (Drummond, 2015, pp.24-26), the mathematician extends the basic pattern of a group in order to form a limit that is not itself realized in any image, representation or exemplification of the mathematical object. In the case of advanced understanding, the mathematician does not see or perceive a group with precise symbols or shapes or forms but perceives an "archetype" (like in "*it's* *an archetype of an element. It is not an element on its own…*" (Q-8)). We evidence here what Desanti implies when he states in (Desanti, 2001, pp.222-224) that "*everything is open in the phenomenological sphere*" and that the manner mathematical objects are grasped in this sphere is "*in tension between two poles*": *i*) a syntactic one that is connected to strict logic (here the functional aspects) and *ii*) a pre-objective or ante-predicative one that precedes logic and that is related to the proper body experience of what is lived (the sensitive and felt aspects unfolded in our study).

Remarks

i) Classical questions of the philosophy of mathematics include the debate between Platonic realism and constructivism. Despite what is evidenced in the present study and the revelation of the sensuous content of the experience of "understanding a mathematical group", there is no direct link that could favor one thesis on the other. For the mathematicians, the abstract object "exists" in the sense that it "is accepted into established mathematics". As emphasized by R. Hersh (Hersh, 2017), mathematical entities have three sides – social, mental, and neural. Mathematicians work on mental entities, manipulate and transform them and it would be incoherent to them to deny that they *are* such mathematical entities. Therefore, our study does not question the existence of ideal objects and their belonging to a specific ontological realm, whether realist or constructivist.

ii) The first remark points at a methodological aspect of our investigation. We did not question the real existence of the mathematical objects the mathematicians speak about. The interview of elicitation, as the psychological phenomenological method (Giorgi, 1985), is performed with the interviewer adopting the psychological phenomenological reduction attitude. It means that during an interview, the researcher brackets everyday or personal knowledge and puts aside presuppositions or theoretical knowledge about the phenomenon under study. She is in a position of openness to the way the phenomenon gives itself to the mathematician. The researcher takes the description of the state-of-affairs as it presents itself to the subject rather than judging its veracity from the objective perspective (for further reading on the psychological reduction see for instance (Finlay, 2008) and Morley (2008/2010)). Therefore, the ontological status of mathematical objects was out of the scope of our investigation.

Acknowledgements

The authors want to thank all the mathematicians who kindly agreed to participate to our study and be interviewed.

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Appendix: GROUP DEFINITION, EXAMPLES, AND DETAILS

A *group* is defined as a set of elements G with an operation * between the elements, satisfying certain axioms:

-- the operation is associative; that is **a*(b*c) = (a*b)*c** for all **a**, **b**, and **c** in G.

-- G has an identity element **1**; that is **a*1 = 1*a = a** for all **a** in G.

-- every element of G has an inverse; that is, for all **a** in G there exists an inverse **a'** such that **a*a' = a'*a = 1**.

For example, the hours on a clock form a group, with the operation of addition that "wraps around" – ten hours plus five hours is three. The nonzero real numbers form a group, with the operation of multiplication. The set of permutations (rearrangements) of a set, such as the different ways to shuffle a deck of cards, forms a group, where the operation is sequential shuffling – one generates a new shuffle by first doing shuffle **a** and then doing shuffle **b**.

The operation in a group, along with the inverses, identity, and associativity, allow for explicit calculation and manipulation of elements of the group. For example, in the clock group, we can calculate the inverse of 4 to be 8, and use the group axioms to prove that inverses are always unique.

Mathematically, every group can, in theory, be described by *generators and relations*. Such a description is called a presentation of the group. The generators are fundamental elements from which every other element can be built. The relations list equations that the generators must satisfy. For example, the clock group is generated by 1 (since 1+1=2, and 1+1+1=3, etc), and the relation that 12 = 0.

One of the most common applications of groups involves a group G *acting* on another set X. Each element of the group permutes the elements of X, and in these interactions mathematicians can learn about both G and X. Sometimes this is described as studying the symmetries of X. For example, there is a certain group that acts on a regular pentagon by rotating it and reflecting it about various axes; this group has ten elements and reveals the different symmetries of the pentagon. In the special case where X is what is called a vector space, mathematicians have developed a rich theory; this is called representation theory, and the actions of G on X are called representations of G.

Within a group, different elements will have different properties, and mathematicians often study different types of elements, developing an elaborate taxonomy. These reveal substructures inside the group, for example by forming *subgroups* within the group. A simple example is the even hours {0,2,4,6,8,10} within the clock group.