What is Subcategory Classification?

a long answer

A "category" is a mathematical world or universe, in which mathematicians ask questions and search for answers. Because math is so abstract, there's often flexibility in choosing what math world we want to work in. You could choose a boring one - for example, look at one or two objects that have no substructure and don't interact. Or you could choose a convoluted, incomprehensible one. But, given the choice, mathematicians usually pick one that is "interesting".

Interesting means not too simple, and not too complicated. It means that there is enough going on - substructures, phenomena, interconnections. And it means there are enough good tools for getting a grasp on these properties.

For example, one type of nice, interesting category is a "co-complete triangulated category". A triangulated category allows you to build triangles out of certain triplets of objects. Using the triangles, you can also sometimes "retract" from one object to another. And being co-complete means that you can take any collection of objects and glue them together in a certain way to get another object in your category.

So suppose we start with an interesting category $\mathcal{C}$. In general, subcategory classification is an attempt to map out the various subcategories in a given category. A subcategory is a sort of self-contained sub-world within your category $\mathcal{C}$.

What does it mean to be self-contained? Suppose $\mathcal{C}$ is a co-complete triangulated category. A "triangulated" subcategory $\mathcal{D}$ of $\mathcal{C}$ is a sub-collection of $\mathcal{C}$ that is self-contained with respect to the operation of making triangles. So if you have a triangle of objects in $\mathcal{C}$, in which two of the objects happen to be in $\mathcal{D}$, then the third must also be in $\mathcal{D}$. A mathematician would call this being "closed under the formation of triangles."

A "thick" subcategory is self-contained, or closed, with respect to forming triangles and taking retracts. A "localizing" subcategory is closed with respect to forming triangles, taking retracts, and gluing objects. (So, if I take a bunch of elements in the localizing subcategory and glue them together, the result is also in the localizing subcategory.)

So you see how having an interesting category - with operations like triangles, retracts, and gluing - yields interesting subcategories. Classifying the different possible subcategories of different categories has yielded some beautiful math. To "classify" means to come up with a good map or description of all the possible subcategories, and how they fit together. For example, you can classify the rooms in a museum by mapping out the different floors, wings, and rooms.

Here's an example of a famous classification. The topological category of Spectra (a spectra is sort of like a space, but can have a negative dimension; see the essay "What are Cohomological Bousfield classes?") is co-complete and triangulated, but extremely complex. If we restrict to so-called "finite" objects, the thick subcategories of this category are beautifully simple - they correspond to and can be classified by all the natural numbers $0, 1, 2, 3, \ldots$.

As another example, the localizing subcategories of the derived category of a Noetherian ring R (see the essay "What is the Derived Category of a non-Noetherian Ring?") correspond to and can be classified by all the subsets of a certain set that is closely related to the ring R.

Let me explain this second example a little more, since its elegance has spurred on and inspired the community of mathematicians studying subcategory classifications. A ring is a mathematical object with a good notion of addition and multiplication. A ring R has a closely related set, $\mathsf{Spec\,R}$, called the prime spectrum of the ring. In a precise sense, the prime spectrum is to its ring as the set of prime numbers is to the integers. Suppose, for the sake of illustration, that you're given a ring R and $\mathsf{Spec\,R}$ has five elements. Suppose you start to write out all the possible subsets of $\mathsf{Spec\,R}$. There are the subsets with only one element - five of these. There are the subsets with two elements each - ten of these. You also have ten different subsets with three elements each, five with four elements each, the unique five element subset ($\mathsf{Spec\,R}$ itself), and the unique empty set.

Suppose you mapped out all these subsets on a piece of paper, and added in arrows to mark when one subset was contained in another. If you organized it well, the picture would be quite nice. (Imagine putting the empty set at the bottom, the full $\mathsf{Spec\,R}$ at the top, and subsets with the same number of elements on the same horizontal line.) What you'd be looking at is called the Boolean algebra of subsets of $\mathsf{Spec\,R}$.

Algebraists were studying a particularly complex category, constructed in an elaborate way from a given ring R. This category is called the derived category of the ring. For a certain type of ring (a "Noetherian" one), they wanted to map out all the possible localizing subcategories. As subcollections of objects in the full category, some subcategories are contained in other subcategories. They found that when they mapped out all the localizing subcategories, and put in arrows to mark when one was contained in another, they arrived at *exactly the same picture* as with the Boolean algebra of subsets of $\mathsf{Spec\,R}$. Not only were the pictures the same, but there was a way of relating the two pictures - there was a one-to-one correspondence between localizing subcategories and subsets of $\mathsf{Spec\,R}$. This is the kind of mathematical result that can make an algebraist's heart race.

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