What is Subcategory Classification?

a technical answer

Let $\mathcal{C}$ be a triangulated category that is complete and co-complete (i.e. all coproducts and products exist). The natural subcollections of $\mathcal{C}$ to study are those that are closed under the various operations that are possible within such a category.

A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *triangulated* if it is closed under the formation of triangles; in other words if $X \to Y \to Z$ is an exact triangle in $\mathcal{C}$ and two of X, Y, and Z are in $\mathcal{D}$, then so is the third.

A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *thick* if it is triangulated and closed under retracts; i.e. if $X \coprod Y$ is in $\mathcal{D}$, then X and Y are in $\mathcal{D}$.

A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *localizing* if it is thick and closed under the formation of arbitrary coproducts; i.e. $\coprod_\alpha X_\alpha$ is in $\mathcal{D}$ for any collection of X_α in $\mathcal{D}$.

A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is *colocalizing* if it is thick and closed under the formation of arbitrary products.

Incidentally, the Eilenberg swindle [HPS97, Sect. 1.4] shows that any subcategory closed under triangles and coproducts is necessarily closed under retracts.

Given some collection A of objects in $\mathcal{C}$, the *thick subcategory generated by* A, denoted $\mathsf{th}\langle A\rangle$, is the intersection of all the thick subcategories containing A. Likewise, we can define the *localizing subcategory generated by* A, denoted $\mathsf{loc}\langle A\rangle$. If X and Y are objects in $\mathcal{C}$ and X is in $\mathsf{loc}\langle Y\rangle$, we say that X *can be built from* Y.

A classification of such subcategories is very useful in practice, because it is often the case that the properties we are interested in are preserved under the formation of triangles, retracts, or coproducts. For example, suppose $\mathcal{C}$ is an axiomatic stable homotopy category, and consider the property P of having homotopy groups of finite type. Since a cofiber sequence in $\mathcal{C}$ yields a long exact sequence of homotopy groups, we see that property P is preserved under the formation of triangles and retracts. If X in $\mathcal{C}$ happens to have homotopy groups of finite type, then for all Y in $\mathsf{th}\langle X\rangle$, we can conclude that Y has homotopy groups of finite type as well.

Besides being useful in understanding the global structure of a category, subcategory classification has yielded some beautiful math.

The first and most famous example is the classification of thick subcategories of finite objects (those in the thick subcategory generated by the sphere object) in the stable homotopy category. These are basically classified by the natural numbers. Every thick subcategory is equal to $\mathcal{C}_n$ for some n, and there is a nested strictly decreasing filtration

$$\cdots \subsetneq \mathcal{C}_{n+1} \subsetneq \mathcal{C}_n \subsetneq \mathcal{C}_{n-1} \subsetneq \cdots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0.$$

Here $C_0 = \mathsf{th}\left\langle S^0\right\rangle = \mathcal{F}$, and for $n \geq 1$, $\mathcal{C}_n = \{X \text{ in } \mathcal{F} : K(n-1)_*(X) = 0\}$, where $K(n)$ is the nth Morava K-theory.

In the algebraic derived category of a commutative Noetherian ring R, Neeman [Nee92] gave a complete classification of localizing subcategories and thick subcategories of finite objects. The former are in one-to-one correspondence with subsets of $\mathsf{Spec}\,\mathsf{R}$, and the latter are in correspondence with the subsets of $\mathsf{Spec}\,\mathsf{R}$ that are closed under specialization.

My approach to studying subcategory classifications is via Bousfield classes. (see the essay "What are Cohomological Bousfield Classes?") Every Bousfield class, homological or cohomological, is a localizing subcategory. It has been conjectured that, in certain categories, every localizing subcategory is also a Bousfield class. This conjecture has been shown to be equivalent to the statement

$$\text{if } \langle X \rangle \leq \langle Y \rangle, \text{ then } X \text{ can be built from } Y.$$

Since Bousfield classes have so much additional structure - in many cases they form a complete lattice, with interesting and well-behaved sublattices - an understanding of Bousfield classes has much to contribute to any study of subcategory classifications.

References

[HPS97] M.Hovey, J.H.Palmieri, and N.P.Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math.Soc. **128** (1997), no. 610, x+114. MR 98a:55017

[Nee92] A. Neeman, *The chromatic tower for D(R)*, Topology **31** (1992), no.3, 519-532.

Luke Wolcott
December 2010
www.forthelukeofmath.com