What is Matrix Algebra?

a long answer

Suppose it takes a plane five hours to fly the 3000 miles from city A to city B, but six hours and fifteen minutes to fly from B to A. The difference is due to a constant wind speed. Assuming the plane travels at a constant speed both ways, what is the plane's speed and the wind speed?

If you assign variables to the unknowns, you can write down two equations. Two equations and two unknowns - then try to solve for the unknowns.

For example, if x is the plane speed and y is the wind speed, we have

$$\begin{aligned} 5(x+y) &= 3000 \\ 6.25(x-y) &= 3000, \end{aligned} \quad \text{so} \quad \begin{aligned} x+y &= 600 \\ x-y &= 480. \end{aligned}$$

Is there always a solution? If there is a solution, is it always unique? What if you have three equations, and three unknowns? Three equations and four unknowns? What about when a delivery company is trying to decide how many vehicles should go from where to where - this problem can have millions of variables and millions of constraining equations!

These questions are the starting point for linear algebra. We want to understand the solutions when we have lots of equations and lots of unknowns. To keep track of all this information, we build a matrix - just a square table of numbers. Once we've organized the information, we can manipulate this matrix in certain ways to get answers to the above questions.

In the example above, we would write

$$\begin{bmatrix} 1 & 1 & 600 \\ 1 & -1 & 480 \end{bmatrix},$$

and using certain manipulations turn this into

$$\begin{bmatrix} 1 & 0 & 540 \\ 0 & 1 & 60 \end{bmatrix},$$

which says the plane speed $x = 540$ mph, and the windspeed $y = 60$ mph.

But once we start playing around with matrices, we find that they're good for lots of other things. There's a way to multiply a matrix C by a matrix D, but unlike with numbers, C times D is not usually the same as D times C. Matrices become interesting mathematical objects in their own right. We can add and multiply them, transpose them, and sometimes invert them. We can ask questions about the properties of matrices and their operations.

And from here, things get really abstract. It turns out we can think of a matrix as a special kind of transformation of space. So studying matrices is the same as studying high-dimensional spaces and transformations between spaces.

For example, 3×3 matrices correspond to transformations of three-dimensional space. The matrix

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

corresponds to rotation ninety degrees up and towards oneself, like drinking a cup. On the other hand, the matrix

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

corresponds to rotation ninety degrees to the right, like a bus driver turning to the right.

Multiplying matrices correspond to putting transformations in sequence. If you rotate as in C, then as in D, this corresponds to the multiplication of the two matrices: $D \times C$. Rotating as in D, then as in C corresponds to $C \times D$. Pick up a book, and rotate C then D, and compare that to rotating D then C. They leave the book in different orientations! This shows physically that $C \times D$ is not the same as $D \times C$!

Mathematicians use matrices to develop elaborate theories about such spacial transformations, but go above three dimensions. These theories actually have many applications, for example to quantum theory and chemistry. Google makes a matrix with billions of rows and billions of columns - one for each website in the internet, and uses linear algebra to calculate its page rankings for web searches.

Matrix algebra, or linear algebra, is a fascinating combination of concrete computations with straightforward arrays of numbers, and mind-bending concepts about higher-dimensional space. It is a relatively accessible jumping-off point for lots of more abstract mathematics. It's one of my favorite classes to teach.

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